The Monotone Integral - Part II (**).

**Sunto.** – Utilizzando la rappresentazione di uno spazio vettoriale localmente convesso E come limite proiettivo di spazi di Banach si estendono risultati ottenuti in [7].

**Summary.** – Using the representation of a lcvs E as the projective limit of Banach spaces we extend the results given in [7].

1. – Introduction.

The aim of this paper is to construct a theory of integration for scalar functions with respect to finitely additive strongly bounded measures with values in a complete locally convex topological vector space. In [6] the Bochner and the monotone integral are compared in nuclear spaces and some relations between them are obtained under suitable conditions concerning the boundedness of the Radon-Nikodym derivative of the measure with respect to any Rybakov control.

Since the definition of the monotone integral is stronger than the definition of Bochner integral, in [7] the authoresses gave another definition of monotone integrability which involves the Mc-Shane integral, in a Banach space X. More precisely, if \( m : \Sigma \rightarrow X \) is a finitely additive bounded measure and \( \lambda \) is the Lebesgue measure, a measurable function

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f : Ω → R^+_0 is integrable in the monotone sense ((*)-integrable) if for every E ∈ Σ, there exists an element w^E ∈ X such that for every ε > 0 there exists a gauge Δ(ε) (which must be the same for every E ∈ Σ) such that

\[ \lim_{n → ∞} \sup \left| w^E - \sum_{i=1}^{n} \lambda(T_i) \phi^E(t_i) \right| ≤ ε \]

for every generalized MeShane partition (T_i, t_i) subordinate to Δ(ε) as defined in [7].

This new definition allows us to obtain the equivalence between Bochner and (✱)-integrals when one integrates with respect to measures taking values in Banach spaces. This paper is the natural continuation of [7]; here in fact the measures with respect to which we integrate take their values in complete locally convex topological vector spaces and the equivalence obtained in [7] follows from the representation of a lcvs E as the projective limit of Banach spaces. Moreover we introduce the weakly (✱)-integral and we compare it with the previous integrals. We observe that in this case to obtain the equivalence between L^{1,*}(m) and (w) − L^1(m) we need a definition of Groetendieck integrability, while in [5, 6] the Bochner integrability (in which the defining sequence does not depend on α ∈ I) was necessary.

2. Definitions of the integrals and their properties.

Let E be a complete locally convex space, m : Σ → E a finitely additive strongly bounded measure.

Every complete locally convex space E is isomorphic to the projective limit of a family of Banach spaces; this family can be chosen such that its cardinality equals the cardinality of a given 0-neighbourhood basis in E (see [9]).

Let \{U_α : α ∈ I\} be a basis of convex and circled neighbourhoods of 0 in E. We say that α ≤ β if U_β ⊆ U_α. If p_α is the gauge of U_α, we can form the projective limit E = π_j(E_α, g_α, β), where E_α = E/U_α is the complete Banach space defined as E_α = E/V_α, where V_α = p_α^{-1}(0), and g_α, β is a continuous linear map of E_β into E_α defined by g_α, β([x]_β) = [x]_α, for every α ≤ β, where [x]_α denotes the equivalence class of the element x with respect to ker (p_α). If α ≤ β then p_α ≤ p_β.

We denote by m_α : Σ → E_α the bounded finitely additive measure defined by m_α(B) = [m(B)]_α for every α ∈ I and for every B ∈ Σ. For every α ∈ I, Σ*,α is the σ-algebra generated by Σ and all m_α-null sets.
We refer to [5] for the notations and definitions relative to each $m_\alpha$ and to [7] for the notion of McShane integral.

Note that, since $m$ is bounded, for every $\alpha \in I$ the semivariation $\|m_\alpha\|$ is bounded as well, and for every $\alpha < \beta$ one easily obtains that $\|m_\alpha\| \leq \|m_\beta\|$.

We consider the following definition of integrability:

**Definition 2.1.** Let $f : \Omega \to \mathbb{R}$ be a $\Sigma$-measurable function. Then $f$ is $m$-integrable iff for every $\alpha \in I$ there exists a sequence of simple functions $(f_n^\alpha)_n$ such that:

1. $(f_n^\alpha)_n\|m_\alpha\|$-converges to $f$, i.e. $f$ is measurable by seminorms;
2. for every $F \in \Sigma$ there exists a $y_F \in E$ such that $\lim_{n \to \infty} p_\alpha (\int_{F} f_n^\alpha \, dm - y_F) = 0$, for every $\alpha \in I$.

In this case we set

$$y_F = \int_{F} f \, dm.$$ 

We denote by $L^1(m)$ the space of $m$-integrable functions.

We observe that if $E$ is a Banach space, $f$ is $m$-integrable iff $f$ is Bochner integrable. So for every $\alpha \in I$ $f$ is Bochner integrable with respect to $m_\alpha$.

In [7] we have introduced the following definition in the case of a Banach space:

**Definition 2.2.** Let $Y$ be a Banach space, $m : \Sigma \to Y$ be a finitely additive bounded measure and $\lambda$ be the Lebesgue measure. A measurable function $f : \Omega \to \mathbb{R}^+$ is integrable in the monotone sense (\((\ast)\)-integrable) if for every $E \in \Sigma$, there exists an element $w^E \in Y$ such that for every $\varepsilon > 0$ there exists a gauge $\Delta(\varepsilon)$ (which must be the same for every $E \in \Sigma$) such that

$$\lim_{n \to \infty} \sup \left| w^E - \sum_{i=1}^{n} \lambda(T_i) \phi^E(t_i) \right| \leq \varepsilon$$

for every generalized McShane partition $(T_i, t_i)$ subordinate to $\Delta(\varepsilon)$.

We denote by $\int_{E} f \, dm = w^E$ and by $L^1(\cdot ; \mathcal{M})$ the space of all \((\ast)\)-integrable functions.
Since for every $\alpha \in I$, $E_\alpha$ is a Banach space and $m_\alpha \colon \Sigma \to E_\alpha$, defined by $m_\alpha(B) = [m(B)]_\alpha$, is a bounded finitely additive measure we can consider the space $L^{1,\ast}(m_\alpha)$ and we prove that:

**Lemma 2.1.** If $f \in \bigcap_{\alpha \in I} L^{1,\ast}(m_\alpha)$ then $\left( \int_{F} f dm_\alpha \right)_{\alpha \in I} \in E$.

**Proof.** If $f \in \bigcap_{\beta \in I} L^{1,\ast}(m_\beta)$, by definition, and Theorem 4.4 of [7], $f \in L^{1,\ast}(m_\beta) = L^1(m_\beta)$, for every $\beta \in I$. We prove that $\left( \int_{F} f dm_\alpha \right)$ is in $E$, namely, for every $\alpha < \beta$,

$$g_{\alpha, \beta} \left( \int_{F} f dm_\beta \right) = \int_{F} f dm_\alpha.$$

By hypothesis $\int_{F} f dm_\beta = \lim_{n \to \infty} \int_{F} f_n^\beta dm_\beta$. Since $f_n^\beta \| m_\beta \|$-converges to $f$, $f$ is $\Sigma^{\ast, \beta}$-measurable. So we have

$$\{ x \in E : |f(x) - f_n^\beta(x)| > t \} \in \Sigma^{\ast, \beta} \subset \Sigma^{\ast, \alpha}$$

and

$$\| m_\alpha \| \| f - f_n^\beta \| > t \| m_\beta \| \| f - f_n^\beta \| > t \|.$$

Therefore $f_n^\beta \| m_\alpha \|$-converges to $f$. Since $(\int_{F} f_n^\beta dm_\beta)_n$ converges in $E_\beta$, it is Cauchy in $E_\beta$; for every $\varepsilon > 0$ there exists $\bar{n} \in \mathbb{N}$ such that for every $r, s > \bar{n}$

$$p_\beta \left( \int_{F} (f_n^\beta - f_s^\beta) dm_\beta \right) < \varepsilon,$$

and so $(\int_{F} f_n^\beta dm_\alpha)_n$ is Cauchy in $E_\alpha$, since

$$\left| \int_{F} (f_n^\beta - f_s^\beta) dm_\alpha \right| = p_\alpha \left( \int_{F} (f_n^\beta - f_s^\beta) dm \right) \leq p_\beta \left( \int_{F} (f_n^\beta - f_s^\beta) dm \right) = \left| \int_{F} (f_n^\beta - f_s^\beta) dm_\beta \right|_\beta.$$

Thus $f$ is $m_\alpha$-integrable since the integral does not depend on the defining sequence $(f_n^\beta)_n$, and we obtain

$$g_{\alpha, \beta} \left( \int_{F} f_n^\beta dm_\beta \right) = \int_{F} f_n^\beta dm_\alpha.$$
and the sequence on the left hand side converges to $\int F dm_a$. Therefore we have

$$g_{a,\beta}\left(\int_F dm_\beta\right) = g_{a,\beta}\left(\lim_{n \to \infty} \int_F f_n^a dm_\beta\right) = \lim_{n \to \infty} g_{a,\beta}\left(\int_F f_n^a dm_\beta\right) = \lim_{n \to \infty} \int_F f_n^a dm_a = \int_F f dm_a.$$ 

Thus $(\int_F f dm_a)_a \in E$, for every $F \in \Sigma$ and

$$g_{a,\beta}\left(\int_F^{*} dm_\beta\right) = g_{a,\beta}\left(\int_F f dm_\beta\right) = \int_F f dm_a = \int_F^{*} f dm_a.$$ 

Now, using the projective structure of $E$ we define the $(\ast)$-integral for a non negative function with respect to a finitely additive strongly bounded measure with values in a complete locally convex space.

**Definition 2.3.** $f : \Omega \to R_0^+$ is $(\ast)$-integrable with respect to $m$ iff $f$ is $(\ast)$-integrable with respect to $m_a$, for every $a \in I$. In this case we set

$$\int_F f dm = \left(\int_F f dm_a\right)_{a \in I}$$

for every $F \in \Sigma$.

We denote by $L^{1,\ast}(m)$ the space of all $(\ast)$-integrable functions.

In [8] the following definition is given:

**Definition 2.4.** $f : \Omega \to R_0^+$ is weakly $(\ast)$-integrable if $\phi : t \mapsto m(f > t)$ is Pettis-integrable and if $t \mapsto |x^* m|(f > t) \in L^1(R_0^+)$, for every $x^* \in E^*$. Note that if $f$ is weakly $(\ast)$-integrable, then for every $F \in \Sigma, f \cdot 1_F$ is weakly $(\ast)$-integrable. In this case we set

$$\int_F f dm = \left(P\right)_{t=0}^{+\infty} m(f \cdot 1_F > t) \, dt.$$ 

If $f$ is real valued we say that $f$ is weakly $(\ast)$-integrable iff $f^+, f^-$ are weakly $(\ast)$-integrable. We denote by $w - L^1(m)$ the space of all weakly $(\ast)$-integrable functions.
3. Comparison.

**Proposition 3.1.** Let \( f : \Omega \to \mathbb{R}_0^+ \) be a measurable function. Then \( f \) is \( m \)-integrable iff \( f \) is \( m_\alpha \)-integrable, for every \( \alpha \in I \).

**Proof.** If \( f \in L^1(m) \) then there exists a sequence of simple functions \((f_n^\alpha)_n\) such that \( f_n^\alpha \| m_\alpha \)-converges to \( f \) and, for every \( F \in \Sigma \), there exists \( y_F \in E \) such that for every \( \alpha \in I \)

\[
\lim_{n \to \infty} p_\alpha \left( \int_F f_n^\alpha \, dm - y_F \right) = 0.
\]

Then,

\[
p_\alpha \left( \int_F f_n^\alpha \, dm - y_F \right) = p_\alpha \left( \left[ \int_F f_n^\alpha \, dm - y_F \right]_\alpha \right) = p_\alpha \left( \left[ \int_F f_n^\alpha \, dm \right]_\alpha - [y_F]_\alpha \right) = p_\alpha \left( \int_F f_n^\alpha \, dm_\alpha - [y_F]_\alpha \right).
\]

So \( f \in \bigcap_{\alpha \in I} L^1(m_\alpha) \).

Viceversa, if \( f \in \bigcap_{\alpha \in I} L^1(m_\alpha) \), then, for every \( \alpha \in I \), there exists a sequence of simple functions \((f_n^\alpha)_n\) which \( m_\alpha \)-converges to \( f \), and such that, for every \( F \in \Sigma \), \( \left( \int f_n \, dm_\alpha \right)_n \) converges in \( E_\alpha \). Fix \( F \in \Sigma \). By Lemma 2, and by Theorem 4.4 of [7], \( \left( \int f_n \, dm_\alpha \right)_n \) is in \( E \). Let \( y_F \in E \) be such that for every \( \alpha \in I \), \( [y_F]_\alpha = \int f \, dm_\alpha \), it only remains to prove that for every \( \alpha \in I, \lim_{n \to \infty} p_\alpha (y_F - \int f_n^\alpha \, dm) = 0 \). But

\[
p_\alpha \left( y_F - \int_F f_n^\alpha \, dm \right) = \left\| \left[ y_F - \int_F f_n^\alpha \, dm \right]_\alpha \right\| = \left\| [y_F]_\alpha - \left[ \int_F f_n^\alpha \, dm \right]_\alpha \right\| = \left\| \int f \, dm_\alpha - \int f_n^\alpha \, dm_\alpha \right\|_\alpha
\]

and so \( f \in L^1(m) \).

**Corollary 3.1.** Let \( f : \Omega \to \mathbb{R}_0^+ \) be a measurable function. Then \( f \) is \( m \)-integrable iff \( f \) is \( (*) \)-integrable with respect to \( m \) and the two integrals agree.
PROOF. It follows from Definitions 2.1 and 2.4, from Lemma 2.3 and from the projective structure of $E$.

Now we want to compare the Bochner integral and the (*) integral with the weak (−)-integral. In order to do this we shall need a preliminary Lemma.

LEMMA 3.2. Let $\alpha \in I$ and let $x_\alpha^* \in E_\alpha^*$. Then $x^*$ defined as $x^*(x) = x_\alpha^*([x]_\alpha)$, belongs to $E^*$.

PROOF. We have only to prove that $x^* = x_\alpha^*pr_\alpha : E \to E_\alpha \to \mathbb{R}$ is continuous at zero. Since $x_\alpha^* \in E_\alpha^*$ then for every $\varepsilon > 0$ there exists $\delta(\varepsilon, \alpha) > 0$ such that for every $y \in E_\alpha$ with $\|y\|_\alpha < \delta$ then $|x_\alpha^*(y)| < \varepsilon$.

Let $V^*_\varepsilon = \{x \in E : p_\alpha(x) \leq \delta(\varepsilon, \alpha)\}$. Let $x \in V^*_\varepsilon$. Then $[x]_\alpha \in E_\alpha$ and $p_\alpha([x]_\alpha) \leq \delta(\varepsilon, \alpha)$. So

$$|x^*(x)| = |x_\alpha^*([x]_\alpha)| < \varepsilon.$$ 

PROPOSITION 3.2. Let $f$ be weakly (−)-integrable with respect to $m$. Then $f$ is weakly (−)-integrable with respect to $m_\alpha$ for every $\alpha \in I$.

PROOF. Let $\alpha \in I$ be fixed. Let $x_\alpha^* \in E_\alpha^*$. By Lemma 3.3 $x^* = x_\alpha^*pr_\alpha \in E^*$. Thus we have

$$x^*(m) = x_\alpha^*([m]_\alpha) = x_\alpha^*(m_\alpha).$$

Hence

$$|x^*(m)|(f > t) = |x_\alpha^*(m_\alpha)|(f > t).$$

Fix $F \in \Sigma$ and set $y_F = (w) - \int \limits_{F} f dm \in E$. Then

$$x^*(y_F) = \int \limits_{0}^{\infty} x^*(m)(f \cdot 1_F > t) \, dt = \int \limits_{0}^{\infty} x^*_\alpha(m_\alpha)(f \cdot 1_F > t) \, dt.$$ 

Therefore $f$ is weakly (−)-integrable with respect to $m_\alpha$, for every $\alpha \in I$ and

$$\int \limits_{[\cdot]} f dm_\alpha = \left[ \int \limits_{[\cdot]} dm \right]_\alpha.$$ 

Now we want to compare $L^{1,*}(m)$ and $w - \widetilde{L}^1(m)$.

First we compare them in a Banach space. Let $Y$ be a Banach space and let $\mu : \Sigma \to Y$ be a finitely additive strongly bounded measure. We set $\phi^y(t) = \mu(\{x \in E : f(x) > t\})$ and $\phi^\mu_y(t) = \mu(\{x \in E : f_\mu(x) > t\})$. 


THEOREM 3.1. If \( f : \Omega \to \mathbb{R}_0^+ \) is a measurable function such that there exists a sequence of simple functions \((f_n)_n\), \(f_n \leq f\) such that

i) \( f_n \|\mu\|\text{-converges to } f\),

ii) \((\omega) - \lim_{n \to \infty} \phi_n^E(t) = \phi^E(t),\)

iii) \(\lim_{n \to \infty} \int_B \phi_n^E(t) \, dt = \int_B \phi^E(t) \, dt\) exists in \(Y\), for the weak topology, for every \(B \in \mathcal{B}\) then \(f\) is \((\ast)\)-integrable with respect to \(\mu\).

PROOF. The proof is the same as in Theorem 4.8 of [7], in fact in order to prove that \(\int \mu(\{x \in E : f_n > t\}) \, dt : \mathcal{B} \to Y\) is countably additive, for every \(E \in \Sigma\), the condition iii) is enough.

THEOREM 3.2. Let \( f : \Omega \to \mathbb{R}_0^+ \) be a measurable function such that \(\lim_{t \to +\infty} \|\mu\|(\{f > t\}) = 0\). Then \(f\) is \((\ast)\)-integrable with respect to \(\mu\) iff \(f\) is weakly \((\ast)\)-integrable with respect to \(\mu\).

PROOF. Suppose that \(f\) is \((\ast)\)-integrable; then for every \(E \in \Sigma\), \(\phi^E(t)\) is McShane-integrable and so by Theorem 1.1 of [4] \(\phi^E(t)\) is Pettis-integrable and the two integrals agree.

It remains to prove that for every \(x^* \in Y^*\) \(|x^* \mu|(\{f > t\}) \in L^1(\mathbb{R}_0^+)\).

Since \(\phi^E(t)\) is Pettis-integrable \(\int_0^{+\infty} |x^* \mu|(\{\omega \in E : f(\omega) > t\}) \, dt < +\infty\) and by Theorems 3.5, 3.6 of [2] \(f\) is integrable with respect to \(|x^* \mu|\).

Suppose now that \(f\) is weakly \((\ast)\)-integrable. By hypothesis for every \(E \in \Sigma\) \(\phi^E\) is Pettis-integrable, namely for every \(E \in \Sigma\) there exists \(w_E \in X\) such that \(\int_0^{+\infty} x^* \mu(\{\omega \in E : f(\omega) > t\}) \, dt = x^* w_E\) and \(|x^* \mu|(f > t) \in L^1(\mathbb{R}_0^+)\).

Since \(f\) is measurable by Proposition 3.2 of [7] \(\phi\) is totally measurable. As in Proposition 3.2 of [7] there exists a sequence of simple functions \((f_n)_n\) with \(f_n \leq f\) for every \(n \in \mathbb{N}\) and \(f_n\) converges to \(f\) \(\mu\)-a.e.

Let \(\phi_n(t) = \mu(\{\omega \in \Omega : f_n(\omega) > t\})\). Then \(\lim_{n \to \infty} \phi_n(t) = \phi(t)\) \(\lambda\)-a.e. and

\[ |x^* \mu|(\{\omega \in \Omega : f_n(\omega) > t\}) \leq |x^* \mu|(\{\omega \in \Omega : f(\omega) > t\}).\]

Let \(E \in \Sigma\) be fixed. We shall prove that the limit \(\lim_{n \to \infty} \int_B \phi_n^E(t) \, dt\) exists in \(Y\), for the weak topology, for every \(B \in \mathcal{B}\).
By Proposition 4.2 of [7] $\phi_n^E$ is Bochner-integrable, for every $n \in \mathbb{N}$ and so $x^* \phi^E_n - x^* \phi_n^E \in L^1(\mathbb{R}^+_0)$. Moreover
\[
| x^* \phi^E_n - x^* \phi_n^E | \leq | x^* \phi^E | + | x^* \phi_n^E | \leq | x^* \mu | \{ \omega \in \Omega : f_n(\omega) > t \} + \\
+ | x^* \mu | \{ \omega \in \Omega : f(\omega) > t \} \leq 2 | x^* \mu | (f > t) \in L^1(\mathbb{R}^+_0)
\]
and so we have
\[
\limsup_{n \to \infty} \int_B | x^* \phi^E_n(t) - x^* \phi_n^E(t) | dt \leq \\
\leq \int_B \limsup_{n \to \infty} | x^* \phi^E_n(t) - x^* \phi_n^E(t) | dt \leq \int_0^{+\infty} | x^* \mu | (f > t) \, dt.
\]
Since $\lim_{n \to \infty} | x^* \phi^E_n(t) - x^* \phi_n^E(t) | = 0$ $\lambda$-a.e. it follows that
\[
\lim_{n \to \infty} \int_B | x^* \phi^E_n(t) - x^* \phi_n^E(t) | dt = 0
\]
and so
\[
\lim_{n \to \infty} \int_B | x^* \phi^E_n(t) dt - x^* w_E | = \lim_{n \to \infty} \int_B | x^* \phi^E_n(t) dt - \int_B x^* \phi^E(t) dt | \leq \\
\leq \lim_{n \to \infty} \int_B | x^* \phi^E_n(t) - x^* \phi^E(t) | dt = 0.
\]
Thus we have proved the existence of the limit $\lim_{n \to \infty} \int_B \phi_n^E(t) dt$ in $Y$, for the weak topology, for every $B \in \mathfrak{B}$.

By Theorem 3.5 $f$ is ($\ast$)-integrable and the two integrals agree.

**Corollary 3.2.** Let $f : \Omega \to \mathbb{R}^+_0$ be a measurable function such that $\lim_{t \to \infty} \|m_a\|(f > t) = 0$. Then $L^1(m) = L^1.(\ast)(m) = w - \tilde{L}^1(m)$.

**Proof.** It follows from Corollary 3.2, Theorem 3.6, where $\mu = m_a$, and Proposition 3.4.

**References**


