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Closure of the Range and Radon-Nikodym Theorems for Vector Finitely Additive Measures with Respect to Different Types of Integration (**).

1. – Introduction.

Let Ω be an arbitrary set, Σ a σ-algebra on Ω and let λ, μ be two non negative finitely additive measures (f.a.m.'s) on Σ, with $\lambda \ll \mu$. Let us suppose that the range of the f.a.m. (μ, λ): $\mathcal{R}(\mu, \lambda)$ is strictly convex (namely its boundary does not contain linear pieces). We shall consider the following three conditions:

$a)$ there exists a μ-integrable function $f: \to \mathbb{R}_0^+$, such that:

$$\lambda(E) = \int_E f\,d\mu \quad \forall E \in \Sigma;$$

$b)$ there exists a family of measurable sets $\{A_r\}_{r>0}$ such that for every $r > 0$ the following three properties hold:

$b1)$ $\lambda(A) - \lambda(B) \geq r[\mu(A) - \mu(B)] \quad \forall B \subset A \subseteq A_r, \; A, B \in \Sigma;$

$b2)$ $\lambda(F) - \lambda(F \cap A_r)) \leq r[\mu(A) - \mu(F \cap A_r)] \quad \forall F \in \Sigma;$

$b3)$ $\lim_{r \to \infty} \lambda(A_r) = 0.$

$c)$ $\mathcal{R}(\mu, \lambda)$ is closed.

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The equivalences between b) and c) and a) and b) have been proven in [4] and [8] respectively. Thus the implications $a) \Leftrightarrow b)$ and $a) \Leftrightarrow c)$ provide two different versions of the Radon-Nikodym theorem in the finitely additive scalar case.

The main idea of this paper is to investigate which ones, among the above equivalences, still hold true when λ ranges on a Banach space. A first difference arising in this case concerns the definition of the integral appearing in a).

Beside the classical integral introduced in [7], recently in [3] Brooks and Martellotti have extended to the vector case the integration theory developed by Greco in [8].

A Radon-Nikodym result for the Dunford-Schwartz integral has been recently proven in [9], while the analogous result for the integral of [3] appeared in [12].

It is worthwhile to observe that, in the investigation relative to the integral in the sense of [3], namely when μ is also a vector f.a.m., it is necessary to equip the Banach space with an order structure in order to extend condition b).

In this paper for each possible implication among conditions a), b) and c) we either prove that the implication holds or we exhibit a suitable counterexample.

Moreover in the appendix some results concerning the geometry of the range of a two-dimensional f.a.m. are given; these properties, which are needed in one counterexample, also complement the H.O.B. property of [10], so characterizing those sets in Σ the measure of which lies on the boundary of the range.

2. – Preliminaries.

Let Ω be a set and Σ a σ-algebra on Ω. Throughout this paper E will denote a Banach lattice with cone C having non empty interior.

DEFINITION 2.1. A f.a.m. $\mu : \Sigma \to C$ is *s-bounded* iff for every sequence $(A_n)_n$ of pairwise disjoint measurable sets

$$\lim_{n\to\infty} \mu(A_n) = 0 .$$

DEFINITION 2.2. If λ and μ are two f.a.m.'s, then λ *is absolutely continuous* with respect to μ, ($\lambda \ll \mu$), iff for every $\varepsilon > 0$ there exists $\delta = \delta(\varepsilon) > 0$ such that $|\lambda|(A) < \delta$ whenever $|\mu|(A) < \varepsilon$, where $|\cdot|$

denotes the total variation in the scalar case and the semivariation in the vector case.

DEFINITION 2.3. A f.a.m. $\nu: \Sigma \to \mathbb{R}_0^+$ is a *control* for a f.a.m. $\mu: \Sigma \to C$ iff $\forall \varepsilon_1, \varepsilon_2 > 0$ there exist $\delta_1, \delta_2 > 0$ such that:

$$\forall E \in \Sigma \quad \text{with} \quad |\mu|(E) < \delta_1 \Rightarrow \nu(E) < \varepsilon_1$$

and

$$\forall E \in \Sigma \quad \text{with} \quad \nu(E) < \delta_2 \Rightarrow |\mu|(E) < \varepsilon_2 .$$

A control is said to be a *Rybakov control* if there exists a linear functional $x^* \in E^*$ such that $\nu = | < x^* | \mu > |$.

The following theorem holds:

THEOREM 2.4 (Rybakov [13]). *If μ is a s-bounded f.a.m., then μ admits a Rybakov-control.*

For every $x^* \in E^*$ $H(x^*, a)$ denotes the hyperplane $\{x \in E: \langle x^* | x \rangle = a\}$. A convex set $K \subset E$ is called *strictly convex* if every supporting hyperplane intersects K in just one point.

Throughout this paper we shall use two different types of integration: we shall give now their definitions.

DEFINITION 2.5 ([7]). Let $\mu: \Sigma \to \mathbb{R}_0^+$ be a f.a.m. A vector function $f: \Omega \to C$ is said to be *integrable* iff there exists a sequence of simple functions $(f_n)_n$ μ-converging to f and such that:

$$\lim_{m, n} \int_\Omega |f_m(X) - f_n(x)| d|\mu| = 0 .$$

We then set:

$$\int f d\mu = \lim_{n \to \infty} \int f_n d\mu .$$

DEFINITION 2.6 ([3]). Let $\mu: \Sigma \to C$ be a s-bounded f.a.m.. A measurable function $f: \Omega \to \mathbb{R}_0^+$ is said to be *μ-integrable* provided there are a control ν and a sequence of simple functions $(f_n)_n$ ν-converging to f and such that the sequence $\left(\int_F f_n d\mu\right)$ converges in E for every F in Σ.

In this case we set:

$$\int f d\mu = \lim_{n\to\infty} \int f_n d\mu .$$

3. - The case of μ scalar.

We begin by observing that when λ is a vector f.a.m., condition *b)* needs to be rewritten. Its suitable version is given in the following theorem.

THEOREM 3.1. *Let $\lambda:\Sigma \to C$ and $\mu:\Sigma\to \mathbb{R}_0^+$ be two f.a.m.'s with $\lambda \ll \mu$. Let $\underline{e} \in \overset{\circ}{C}$ be fixed and let $D=\{i/2^n,\, i,\, n\in \mathbb{N}\}$. If a decreasing family of Σ-measurable sets $\{A_r\}_{r\in D}$ exists such that $A_0 = \Omega$ and for every $r>0$ the following relationships hold:*

b1') $\lambda(A)-\lambda(B) \geqslant r\underline{e}[\mu(A)-\mu(B)] \quad \forall B\subset A \subseteq A_r;\ \ A, B\in\Sigma;$

b2') $\lambda(F)-\lambda(F\cap A_r) \leqslant r\underline{e}[\mu(A)-\mu(F\cap A_r)] \quad \forall F\in\Sigma;$

b3) $\lim_{r\to\infty} \lambda(A_r) = 0;$

then a) holds and the map $f:\Omega\to C$ defined by $f(x)=\sup\{r\in D\colon x\in A_r\}\cdot \underline{e}$ is the required Radon-Nikodym derivative.

PROOF. The proof is analogous to that in [8] as soon as one proves that, for every $F\in\Sigma$, $\int_F f d\mu = \lim_{n\to\infty} \int_F f_n d\mu$, where:

$$f_n(x) = \frac{1}{2n}\cdot \underline{e} \sum_{k=1}^{n2^n} 1_{A_{k/2n}} .$$

In order to do this let us put:

$$g(x)=\sup\{r\in D\colon x\in A_r\},$$

$$g_n(x) = \frac{1}{2n} \sum_{k=1}^{n2^n} 1_{A_{k/2n}} .$$

By Lemma 2 of [8] g_n converges to g in $L^1(\mu)$, and therefore setting $G_{n,\varepsilon} = \{x\colon |g_n(x)-g(x)|>\varepsilon\}$,

$$\lim_{n\to\infty} \mu(G_{n,\varepsilon}) = 0 \quad \forall \varepsilon > 0 .$$

Let us set $H_{n,\sigma} = \{x\colon \|f_n(x)-f(x)\| > \sigma\}$. Since $\underline{e}$ is an order unit there will be a $k > 0$ such that $k\underline{e} > \sigma$ and so

$$\{x\colon \|f_n(x) - f(x)\| > \sigma\} \subset \{x\colon |g_n(x) - g(x)| > k\}.$$

Hence $H_{n,\sigma} \subset G_{n,k}$. In order to prove that f_n μ-converges to f we have to show that

$$\lim_{n\to} \inf_{\alpha>0} \arctan(\alpha + |\mu|(\{x\colon \|f_n(x) - f(x)\| > \alpha\})) = 0 .$$

As $|\mu|(\cdot) \leqslant 4|\mu(\cdot)|$ and α goes to zero the above assertion is easily verified. It now remains to prove that $\lim_{m,n\to\infty} \int_\Omega \|f_m(x) - f_n(x)\|\, d\mu = 0$. To prove this it suffices to apply [8] Main Theorem once noticed that $\|f_n - f_m\| = |g_n - g_m| \cdot \|\underline{e}\|$.

In this way we have proven the implication $a) \Rightarrow b)$. To show that the converse is not true we shall now provide a suitable counterexample.

EXAMPLE 3.2. Let $\Omega, \Sigma, \mu\colon \Sigma \to \mathbb{R}_0^+$, $\nu\colon \Sigma \to \mathbb{R}^2$, $\nu \geqslant 0$ be such that $R(\mu,\nu)$ is a strictly convex set of $\mathbb{R}^3$ having non-empty interior (observe that it is always possible to choose Ω, Σ, μ, ν such that these conditions are satisfied). If a density exists, as (μ,ν) is three-dimensional, according to [4] $\mathcal{R}(\mu,\nu)$ is closed. Assume by contradiction that property $b)$ holds, namely:

- for $\underline{e} = (e_1, e_2) \in \mathring{C}$ where $C = (\mathbb{R}_0^+)^2$ there exists a family of sets $\{A_r\}_{r\in D} \subset \Sigma$ such that $\forall r > 0$ the following statements hold:

$b1')$ $\nu_i(H) \geqslant re_i\mu(H) \quad \forall H \in \Sigma,\ H \subset A_r$;

$b2')$ $\nu_i(H) \leqslant re_i\mu(H) \quad \forall H \in \Sigma,\ H \subset A_r^c$;

$b3)$ $\lim_{r\to\infty} \nu(A_r) = 0$.

By Lemma 5.2 the set $(e_i\nu, \nu_i)(A_r) \in \partial\mathcal{R}(e_i\mu, \nu_i)$ for all $r > 0$ and $i = 1, 2$. Then by Lemma 5.3 $(\mu, \nu_i)(A_r) \in \partial\mathcal{R}(\mu, \nu_i)$ for every $r > 0$ and $i = 1, 2$ which is impossible.

The following recent result shows the implication $a) \Rightarrow c)$ under suitable conditions.

THEOREM 3.3 (Bassi-Gamba [1]). *Let Ω be a set, Σ a σ-algebra on Ω and $f\colon \Omega \to C$ a Σ-measurable function. Let $\mu\colon \Sigma \to \mathbb{R}_0^+$ be a f.a.m. If for the set $\mathcal{R} = \left\{\int_F f\,d\mu;\ F \in \Sigma\right\}$ the following properties are satisfied:*

3.3.1) $\overline{\mathcal{R}}$ *is weakly bounded and strictly convex;*

3.3.2) $\overset{\circ}{\mathcal{R}} \neq \emptyset$;

3.3.3) $\partial\mathcal{R} = \partial\overline{\mathcal{R}}$;

then $\mathcal{R}$ *is closed.*

From this theorem it follows that property c), even under the conditions of Theorem 3.3, does not imply property b).

Radon-Nikodym results of the type $c) \Rightarrow a)$, under suitable conditions, are given in [5].

We prove that, in the general case, the implication $c \Rightarrow a$) fails to be true.

EXAMPLE 3.4. Let $\Omega = [0,1]$, $\mathcal{B}$ be the Borel σ-algebra on Ω, $\mu : \mathcal{B} \to [0,1]$ be the Lebesgue measure and λ: $\mathcal{B} \to L^1(\mu)$ be the f.a.m. defined by $\lambda(E) = 1_E$. It is known [6] that the Radon-Nikodym derivative $d\lambda/d\mu$ does not exist although λ is absolutely continuous with respect to μ. We prove now that $\mathcal{R}(\lambda, \mu)$ is closed.

Let $(\lambda(E_n), \mu(E_n))_n$ be a sequence converging to (f, x) with $f \in L^1(\mu)$, $x \in [0,1]$. (The convergence is that of $L^1(\mu) \times \mathbb{R}_0^+$). Since 1_{E_n} converges to f in L^1 then 1_{E_n} μ-converges to f and therefore there exists a subsequence $1_{E_{n_k}}$ which converges to f almost everywhere. So μ-a.e. f ranges in $\{0,1\}$. Namely, setting $E_0 = \{x \in [0,1]: f(x) = 1\}$, 1_{E_0} is a representative of f in $L^1(\mu)$. Then 1_{E_n} converges to 1_{E_0} in L^1. By

$$\|1_{E_n} - 1_{E_0}\|_1 = \int_\Omega |1_{E_n} - 1_{E_0}|\, d\mu = \int_\Omega 1_{E_n \Delta E_0}\, d\mu = \mu(E_n \Delta E_0)$$

we have $\mu(E_n \Delta E_0) \to 0$ and therefore $\mu(E_n) \to \mu(E_0)$.

So

$$(f, x) = (1_{E_0}, \mu(E_0)) = (\lambda(E_0), \mu(E_0)) \in \mathcal{R}(\lambda, \mu).$$

Nevertheless the implication $c) \Rightarrow a)$ is true in the finite dimensional case as the following theorem shows:

THEOREM 3.5. *Let* $E = \mathbb{R}^n$ *and suppose that* $\mathcal{R}(\lambda, \mu)$ *is closed. Then there exists a* μ*-integrable function* f: $\Omega \to \mathbb{R}^n$ *such that:*

$$\lambda(E) = \int_E f\, d\mu \qquad \forall E \in \Sigma.$$

PROOF. By the closedness of the range the closure of each projection $\mathcal{R}(\lambda_i, \mu)$ follows for every $i = 1, \ldots, n$. As in the scalar case the theorem is true ([4]), for every $i = 1, \ldots, n$ there exists f_i which is the Radon-Nikodym derivative for the pair (λ_i, μ). The vector $f = (f_1, \ldots, f_n)$ is the required derivative.

4. – The case of μ vector.

In this section we assume that μ is C-valued and s-bounded; moreover, for each $x^* \in C^*$ we set $\mu^* = \langle x^* \mu \rangle$. In order to prove the equivalence $a) \Leftrightarrow b)$ we introduce some preliminary lemmata.

LEMMA 4.1. *Let $f: \Omega \to \mathbb{R}_0^+$ be a μ-integrable function; then $\int_F f d\mu \in C$ for every $F \in \Sigma$.*

PROOF. For $x^* \in C^*$ fixed, we have

$$0 \leqslant \int_F f d\mu^* = \left\langle x^* \middle| \int_F f d\mu \right\rangle .$$

Being $\mathring{C} \neq \emptyset$, $\bigcap_{x^* \in C^*} F(x^*, 0) = C$, where $F(x^*, 0) = \{x \in E : \langle x^* | x \rangle \geqslant 0\}$; it follows that $\int_F f d\mu \in C$.

LEMMA 4.2. *Let $\lambda, \mu : \Sigma \to C$ be two f.a.m.'s with $\lambda \ll \mu$. If there exists a Σ-measurable function $f: \Omega \to \mathbb{R}_0^+$ such that $\lambda(F) = \int_F f d\mu$ for every $F \in \Sigma$ then the family of sets $A_r = \{x \in \Omega : f(x) > r\}$ satisfies the following properties:*

$b1)$ $\lambda(A) - \lambda(B) \geqslant r[\mu(A) - \mu(B)] \quad \forall A, B \in \Sigma, \;\; B \subset A \subsetneq A_r$;

$b2)$ $\lambda(F) - \lambda(F \cap A_r) \leqslant r[\mu(A) - \mu(F \cap A_r)] \quad \forall F \in \Sigma$;

$b3')$ $(w) - \lim_{r \to \infty} \lambda(A_r) = 0$.

PROOF. $A_r \in \Sigma$ for every $r \in \mathbb{R}_0^+$, and by Lemma 4.1 properties $b1)$ and $b2)$ hold. In fact, setting $H = A \setminus B$, we find,

$$\lambda(H) = \int_H f d\mu \geqslant \int_H r \, d\mu = r\mu(H)$$

since $\int_H (f - r) \, d\mu \in C$. We analogously show that $b2)$ holds. Now, for $x^* \in C^*$ fixed, suppose by contradiction that $\lim_{n \to \infty} \mu^*(A_n) = k$, with $k > 0$

(such limit exists because the sequence $\mu^*(A_n)$ is monotonic); then

$$\int_\Omega f d\mu^* \geqslant \int_{A_n} f d\mu^* \geqslant n\mu^*(A_n) \quad \forall n \in \mathbb{N}$$

and therefore

$$\int_\Omega f d\mu^* \geqslant \lim_{n\to\infty} n\mu^*(A_n) = +\infty .$$

Then f is not μ^*-integrable and hence it is not μ-integrable. This is impossible and therefore $\lim\limits_{n\to\infty} \mu^*(A_n) = 0$ for every $x^* \in C^*$, namely $(w) - \lim\limits_{n\to\infty} \mu(A_n) = 0$ and by the absolute continuity of λ with respect to μ property $b3'$) follows.

LEMMA 4.3. *Let $\lambda, \mu : \Sigma \to C$ be two f.a.m.'s with $\lambda \ll \mu$. Set $D = \{i/2^n,\, i, n \in \mathbb{N}\}$. If a decreasing family of sets $\{A_r\}_{r\in D}$ exists, such that $A_0 = \Omega$ and b1), b2), b3') of* Lemma 4.2 *are satisfied, then the function $f\colon \Omega \to \mathbb{R}_0^+$ defined by*

$$f(x) = \sup\{r \in D : x \in A_r\}$$

is the Radon-Nikodym derivative, i.e. for every $F \in \Sigma$ $\lambda(F) = \int_F f d\mu$.

PROOF. By Lemma 2 of [8] for every $x^* \in C^*$ we have that f is μ^*-integrable and $\lambda^*(F) = \int_F f d\mu^*$.

We shall show that f is a μ-integrable function such that for every $F \in \Sigma$, $\lambda(F) = \int_F f d\mu$.

For every $n \in \mathbb{N}$ we consider the simple function

$$f_n(x) = \frac{1}{2n} \sum_{k=1}^{n2^n} 1_{A_{k/2^n}}(x)$$

and the function

$$g_n = f \wedge n - f \wedge \frac{1}{2n} .$$

We also put $F_n = \{x \in \Omega : f(x) > n\}$, $G_n = \{x \in \Omega : f(x) > 1/2n\}$ and for every $\alpha > 0$ fixed, $H_n(\alpha) = \{x \in \Omega : |g_n(x) - f(x)| > \alpha\}$. Then $H_n \subset F_n \subset G_n$. In exactly the same way as done in Lemma 2 of [8] one can show the following assertion:

4.3.1) $g_n \leqslant f_n \leqslant f$;

4.3.2) $\lim_{n\to\infty} \mu^*(H_n(\alpha)) = 0 \quad \forall x^* \in C^*, \ \forall \alpha > 0$;

4.3.3) $\lim_{n\to\infty} \int_F f_n d\mu^* = \int_F f d\mu^*$.

Let $\bar{x}^* \in C^*$ be such that $\bar{\mu}^* = |\langle \bar{x}^* | \mu \rangle|$ is a Rybakov control for μ. We shall prove that the sequence $(f_n)_n$ converges to f in $\bar{\mu}^*$-measure and that for every $F \in \Sigma$ the sequence $\left(\int_F f_n d\mu \right)_n$ is Cauchy in C.

From 4.3.1) above, $K_n(\alpha) = \{x\colon |f_n(x) - f(x)| > \alpha\} \subset H_n(\alpha)$; hence from 4.3.2), $\lim_{n\to\infty} \mu^*(K_n(\alpha)) = 0$ for every $x^* \in C^*$.

In particular $\lim_{n\to\infty} \bar{\mu}^*(K_n(\alpha)) = 0$ for every $\alpha > 0$. In order to prove that, for every $F \in \Sigma$, the sequence $\left(\int_F f_n d\mu \right)_n$ is Cauchy in C, let $\varepsilon > 0$ and $F \in \Sigma$ fixed, and let $E_n = \left\{ |f - f_n| \geqslant (\varepsilon / 4 |\mu|(\Omega)) \right\}$. It is possible to find $\bar{n}(\varepsilon)$ in such a way that, for every $n > \bar{n}$, $1/2n < \varepsilon/4|\mu|(\Omega)$ and the inclusions $E_n \subset H_n \subset F_n$ hold.

Let $\bar{\lambda}^* = |\langle \bar{\bar{x}}^* | \lambda \rangle|$ be a Rybakov control for λ (observe that, since $\lambda \ll \mu$, and μ is s-bounded, λ is s-bounded and thus Theorem 2.4 ensures that such a control exists). Let $\sigma = \sigma(\varepsilon/8)$ be such that $\bar{\lambda}^*(E) < \sigma$ implies $|\lambda|(E) = \sup_{x^* \in B_1^*} |\langle x^* | \lambda \rangle| < \varepsilon/8$. Form $b3'$) we can pick an integer $\bar{n}(\varepsilon/8) > \bar{n}$ such that $\bar{\lambda}^*(E_n) < \sigma$ for every $n > \bar{n}$.

Then for every $n, m > \bar{n}$, taking into account the inequality 4.3.1) and the above considerations, we have:

$$\left\| \int_F (f_n - f_m)\, d\mu \right\| = \sup_{x^* \in B_1^*} \left| \int_F (f_n - f_m)\, d\mu^* \right| \leqslant$$

$$\leqslant \sup_{x^* \in B_1^*} \left| \int_F (f_n - f)\, d\mu^* \right| + \sup_{x^* \in B_1^*} \left| \int_F (f - f_m)\, d\mu^* \right| \leqslant$$

$$\leqslant \sup_{x^* \in B_1^*} \left| \int_{F \cap E_n} (f_n - f)\, d\mu^* \right| + \sup_{x^* \in B_1^*} \left| \int_{F - E_n} (f_n - f) d\mu^* \right| +$$

$$+ \sup_{x^* \in B_1^*} \left| \int_{F \cap E_m} (f - f_m)\, d\mu^* \right| + \sup_{x^* \in B_1^*} \left| \int_{F - E_m} (f - f_m) d\mu^* \right| \leqslant$$

$$\leqslant \sup_{x^*\in B_1^*}\left|\int_{E_n}(f_n-f)\,d\mu^*\right| + \sup_{x^*\in B_1^*}\frac{\varepsilon}{4|\mu|(\Omega)}|\mu^*|(\Omega) +$$

$$+ \sup_{x^*\in B_1^*}\left|\int_{E_m}(f-f_m)\,d\mu^*\right| + \sup_{x^*\in B_1^*}\frac{\varepsilon}{4|\mu|(\Omega)}|\mu^*|(\Omega) \leqslant$$

$$\leqslant \frac{\varepsilon}{2} + \sup_{x^*\in B_1^*}\int_{E_n}|f_n|d|\mu^*| + \sup_{x^*\in B_1^*}\int_{E_n}|f|d|\mu^*| +$$

$$+ \sup_{x^*\in B_1^*}\int_{E_m}|f|d|\mu^*| + \sup_{x^*\in B_1^*}\int_{E_m}|f_m|d|\mu^*| \leqslant$$

$$\leqslant \frac{\varepsilon}{2} + 2\cdot\sup_{x^*\in B_1^*}\int_{E_n}|f|d|\mu^*| + 2\cdot\sup_{x^*\in B_1^*}\int_{E_m}|f|d|\mu^*| =$$

$$= \frac{\varepsilon}{2} + 2\cdot\sup_{x^*\in B_1^*}|\lambda^*|(E_n) + 2\cdot\sup_{x^*\in B_1^*}|\lambda^*|(E_m) \leqslant \frac{\varepsilon}{2} + 2\cdot\frac{\varepsilon}{8} + 2\cdot\frac{\varepsilon}{8} = \varepsilon.$$

Then from relationship (4.3.3)

$$\lim_{n\to\infty}\int_F f_n\,d\mu = \int_F f\,d\mu\,.$$

THEOREM 4.4. *Let $\lambda, \mu : \Sigma \to C$ be a pair of f.a.m. with $\lambda \ll \mu$. The following conditions are equivalent:*

a) there exists a Σ-measurable function $f\colon \Omega \to \mathbb{R}_0^+$ such that $\lambda(F) = \int_F f\,d\mu$ for every $F \in \Sigma$;

b′) there exists a family of Σ-measurable sets $\{A_r\}_{r\in D}$ such that $\forall r > 0$ the following statements hold:

b1) $\lambda(A) - \lambda(B) \geqslant r[\mu(A) - \mu(B)] \quad \forall B \subset A \subseteq A_r,\ A, B \in \Sigma;$

b2) $\lambda(F) - \lambda(F \cap A_r) \leqslant r[\mu(A) - \mu(F \cap A_r)] \quad \forall F \in \Sigma;$

b3′) $(w)\text{-}\lim_{r\to\infty} \lambda(A_r) = 0.$

PROOF. The same as in [8].

In [1] it is proven that, in the vector case, and under suitable conditions, property *a*) implies property *c*). In the sequel we give the analogous theorem in the case of the integration introduced in [3].

THEOREM 4.5. *Let Ω be a set, Σ an algebra on Ω and $f\colon \Omega \to \mathbb{R}_0^+$*

a Σ-measurable function. Let $\mu:\Sigma\to C$ be a f.a.m. If the set $\mathcal{R} = \left\{ \int_F f d\mu;\ F\in\Sigma \right\}$ *satisfies:*

4.5.1) $\overline{\mathcal{R}}$ *is strictly convex;*

4.5.2) $\overset{\circ}{\mathcal{R}} \neq \emptyset$;

4.5.3) $\partial\mathcal{R} = \partial\overline{\mathcal{R}}$;

then $\mathcal{R}$ *is closed.*

Proof. The proof is substantially analogous to that of [1]; it suffices to observe that, by the s-boundedness of μ, $\mathcal{R}(\mu)$ is bounded [10].

Conversely the closure of the range does not imply, in the general case, the existence of a Radon-Nikodym derivative, as the following Example shows.

Example 4.6. Let $\Omega=[0,1]$, Σ be the Borel σ-algebra and m_1 be the Lebesgue measure. We consider the functions $f_1, f_2:\Omega\to\mathbb{R}_0^+$ defined by:

$$f_1(x)=x \qquad f_2(x)=1-x\,.$$

Let m_2, m_3 be the measures defined by:

$$m_2(\cdot)=\int_{\cdot} f_1\,dm_1 \qquad m_3(\cdot)=\int_{\cdot} f_2\,dm_2\,.$$

We put $m=((m_2, m_1);(m_3, m_2))$. By construction $(m_3, m_2)\ll(m_2, m_1)$. Moreover, since the m_i's are countably additive, the range of m is closed by virtue of the Liapounov property. We shall show that no Radon-Nikodym derivative exists, for the pair of measures (m_3, m_2), (m_2, m_1). In fact if, by contradiction, there exists a Σ-measurable function $f:\Omega\to\mathbb{R}$ such that:

$$(m_3(A), m_2(A)) = \int_A f d(m_2, m_1) = \left(\int_A f dm_2;\ \int_A f dm_1\right) \qquad \forall A\in\Sigma$$

then

$$\left(\int_A f dm_2;\ \int_A f dm_1\right) = \left(\int_A f_2\,dm_2;\ \int_A f_1\,dm_1\right) \qquad \forall A\in\Sigma$$

and therefore $f=f_2$ m_2-a.e. and $f=f_1$ m_1-a.e.; thus, $f=f_1$ m_2-a.e. and henceforth $f_1=f_2$ m_2-a.e. which is impossible.

It follows immediately that the closure of the range, under the hypothesis of strict convexity, does not imply property *b*). Indeed, if the implication held, from the «only if» part of Theorem 4.4 and from Theorem 4.5 it would follow that the closure of the range implies the existence of a Radon-Nikodym derivative.

5. – Appendix.

LEMMA 5.1. *Let $f:[a,b]\to\mathbb{R}_0^+$ be a monotone concave function. Let $x_0\in\,]a,b[$ and $\alpha>0$; if s denotes the left hand side derivative: $s=f_s'(x_0)$, then for every $r\leqslant s$ and for every pair (x,y) such that $y/x\leqslant r$, and $0<x\leqslant x_0-a\leqslant x_0$ it follows $f(x_0)+\alpha-y>f(x_0-x)$.*

PROOF. f being concave, one finds

$$\frac{f(x_0)-f(x_0-x)}{x}\geqslant f_s'(x_0)=s\geqslant r\geqslant\frac{y}{x}$$

and, since $x>0$

$$f(x_0)-f(x_0-x)\geqslant y\,,$$

that is

$$f(x_0)-y\geqslant f(x_0-x)\,.$$

Thus, as $\alpha>0$,

$$f(x_0)+\alpha-y\geqslant f(x_0-x)$$

LEMMA 5.2. *Let $m=(m_1,m_2)$ be a f.a.m. For every $r>0$ the family of sets $\{A_r\}_r$ introduced in property b) is such that $m(A_r)\in\partial R(m)$.*

PROOF. Assume by contradiction that there exists a real number $r>0$ such that $m(A_r)\in\overset{\circ}{\mathcal{R}}(m)$. Then there is a $H\in\Sigma$ such that $m_1(H)=m_1(A_r)$ but $m_2(H)>m_2(A_r)$. Let us set $f=G_{A_r}:[0,m_1(A_r)]\to\mathbb{R}_0^+$ where $G_{A_r}(x)=\sup\{y:(x,y)\in\overline{\mathrm{co}}(\mathcal{R}(m|_{A_r}))\}$. As in [2], G_{A_r} is continuous, non decreasing and concave, with value 0 at 0. Setting

$$x_0=m_1(H)=m_1(A_r),\qquad \alpha=m_2(H)-m_2(A_r),$$
$$x=m_1(H\cap A_r^c),\qquad y=m_2(H\cap A_r^c),$$

we find $G_{A_r}(x_0) = m_2(A_r)$, and, by the monotonicity, $x = m_1(H \cap A_r^c) \leqslant \leqslant m_1(H) = x_0$. Moreover from $b2)$ $m_2(H \cap A_r^c) \leqslant rm_1(H \cap A_r^c)$. Then from the previous Lemma,

$$m_2(A_r) + m_2(H) - m_2(A_r) - m_2(H \cap A_r^c) > G_{A_r}(m_1(H) - m_1(H \cap A_r^c)),$$

namely,

$$m_2(H \cap A_r) > G_{A_r}(m_1(H \cap A_r))$$

which contradicts the definition of G_{A_r}.

This Lemma, together with the H.O.B. property of[11] gives a geometric characterization of those sets H such that $m(H) \in \partial^- \mathcal{R}(m)$. They are in fact exactly those for which a property of subrange separation holds, that is those for which there exists just one $r > 0$ such that: for every $A \subset H$ $m_2(A) \leqslant rm_1(A)$ and for every $A \subset H^c$ $m_2(A) \geqslant \geqslant rm_1(A)$.

LEMMA 5.3. *Let* $\pi = (\pi_1, \pi_2)\colon \Sigma \to \mathbb{R}^2$ *be a non negative f.a.m. Let* $x > 0$, *and* λ *denote the f.a.m.* $\lambda = (x\pi_1, \pi_2)$; *then for every* A *such that* $\pi(A) \in \partial^- R(\pi)$, $\lambda(A) \in \partial^- R(\lambda)$.

PROOF. Obvious.

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