Remarks on set valued integrals of multifunctions with non empty, bounded, closed and convex values*

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Mathematical Subject Classification: **28B20**, **26E25**, **54C60**, **54C65**, **46G10**

Keywords: Multivalued functions, measurable selections, integrable boundedness, Hausdorff metric, Aumann integral, Bochner integral, totally measurable multifunctions.

Abstract: We study the comparison between the Aumann and the Bochner integrals for integrably non empty bounded closed convex valued multifunctions in a separable Banach space when it is not possible to apply embedding theorems.

1 Introduction

The study of measurable multifunctions has been developed extensively with applications to mathematical economics and optimal control theory by many authors. The natural approach, which derives from the study of the integro differential inclusions, is due to Aumann in 1965 ([2]) and is

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based on the integration of measurable selections.

Unfortunately the Aumann integral does not satisfy all the usual properties of an integral. So it seems to be natural to investigate whether the Aumann integral can be regarded as a Bochner or Debreu integral, [5].

Hiai and Umegaki in [9] and Byrne in [3], using the classical Rädstrom embedding theorem (see [12]), consider an integrable multifunction as a vector function, but to obtain this they have to take only integrable multifunctions with convex and compact or weakly compact values; in the second case, moreover, Byrne requires also that the multifunction is the limit of a sequence of measurable, simple multifunctions since the space of all convex weakly compact subsets of a Banach space X is in general not separable.

In [10] and [13] the authors introduced the Bochner integral for multifunctions, with values in non empty, bounded, closed and convex subsets of a locally convex topological vector space, with respect to finitely additive measures.

In [11] the Authors compare the Bochner and Aumann integrals, for compact convex valued multifunctions, in the finitely additive setting by making use of the results obtained by Hiai and Umegaki in [9] and the Stone transform which allows to obtain the results in finitely additive setting via the same results in the countable additive setting.

Here we consider the space $cb(X)$ of all non empty, bounded, closed, convex subsets of a separable space X and we compare the Bochner integral, introduced in [10] and [13], with the Aumann integral when the Rädstrom embedding theorem, used in [9], [3] and [11], fails to be true.

In this paper we obtain coincidence results between the two integrals for totally measurable multifunctions F when X^* has the Radon-Nikodym property.

The Author would like to thank Professor Candeloro and Professor Martellotti for their helpful suggestions and for their patience in reading several versions of this paper.

2 Notations and Preliminaries

We will use the following definitions and notations.

_ X is a separable Banach space.

_ X^* is the topological dual of X, X^*_s (resp. X^*_b) is the vector space X^* equipped with the weak* $\sigma(X^*, X)$ (resp. norm) topology.

_ $\overline{B}_X$ (resp. $\overline{B}_{X^*}$) is the closed unit ball in X (resp. X^*_b).

_ $cb(X)$ is the collection of all non empty convex closed bounded subsets of X.

_ The addition $\dot{+} : cb(X) \times cb(X) \to cb(X)$ is defined as follows : $A \dot{+} B = cl\{A + B\}$.

The addition $\dot{+}$satisfies the following properties: if $A, B \in cb(X)$ and λ is a real number

$$(A \dot{+} B) \dot{+} C = A \dot{+} (B \dot{+} C), \quad A \dot{+} B = B \dot{+} A,$$

$$\lambda(A \dot{+} B) = \lambda A \dot{+} \lambda B, \quad \lambda_1(\lambda_2 A) = (\lambda_1 \lambda_2)A, \quad 1A = A$$

Moreover if $A \dot{+} C = B \dot{+} C$ then $A = B$. (*Cancellation law*)

_ If $A_i, i = 1, \cdots, n$ are in $cb(X)$ we denote by $\sum_{i=1}^n A_i$ the set $A_1 \dot{+} \cdots \dot{+} A_n$.

_ $\delta^*(x', A) = \sup\{< x', x >: x \in A\}$ is the support function of a subset $A \subset X$.

_ If A and B are subsets of X, the excess of A over B is

$$e(A, B) = \sup\{d(a, B) : a \in A\}$$

and the Hausdorff distance between A and B is

$$h(A, B) = \max\{e(A, B), e(B, A)\}.$$

For the properties of h see Proposition 1.2 of [10].

_ The excess $e(A, \{0\}) = h(A, \{0\})$ is denoted by $|A|$. Then

$$|A| = \sup\{\|a\| : a \in A\}.$$

_ (Ω, Σ, μ) is a probability space.

_ A multifunction $F : \Omega \to cb(X)$ is *Effros-measurable* (shortly-measurable) if the set $F^- U = \{\omega \in$

$\Omega : F(\omega) \cap U \neq \emptyset\}$ belongs to Σ for any open subset U of X.

_ $\mathcal{L}^1_{cb}(X)$ is the subspace of all the measurable multifunctions $F : \Omega \to cb(X)$ such that $|F| \in L^1_{I\!R^+}$, where two multifunctions F_1 and F_2 are considered identical if there exists a μ-null set N such that, for every $\omega \notin N$, $F_1(\omega) = F_2(\omega)$.

_ A measurable multifunction $F : \Omega \to cb(X)$ is said to be *integrably bounded* if there exists an integrable non negative function $g : \Omega \to I\!R^+_0$ such that for a.e. $\omega \in \Omega$, $|F(\omega)| \leq g(\omega)$. If F is measurable and integrably bounded then $F \in \mathcal{L}^1_{cb}(X)$.

_ It is known that $F \in \mathcal{L}^1_{cb}(X)$ if and only if the set of all the Bochner integrable selections of F, denoted by S^1_F, is non empty and bounded in $L^1(\mu, X)$ (see for example Theorem 3.2 of [9]).

_ The Aumann integral $(A) - \int_E F d\mu$ of F over a measurable set E is given by

$$(A) - \int_E F d\mu := \left\{ \int_E f d\mu : f \in S^1_F \right\}.$$

_ $\mathcal{E}^1_{cb}(X)$ is the subset of all the simple multifunctions $F : \Omega \to cb(X)$. One has $\mathcal{E}^1_{cb}(X) \subset \mathcal{L}^1_{cb}(X)$.

_ A measurable multifunction $F : \Omega \to cb(X)$ is *totally measurable* if there exists a sequence of simple multifunctions (F_n) with values in $cb(X)$ such that $\lim_{n\to\infty} h(F_n(\omega), F(\omega)) = 0$ for a.e. $\omega \in \Omega$.

_ We denote by $\mathcal{M}^1_{cb}(X)$ the subspace of $\mathcal{L}^1_{cb}(X)$ of all totally measurable and integrably bounded multifunctions.

If F is totally measurable and single valued, by definition the range of F is separable.

When dealing with the multivalued case, in [9] an example is given (Example 3.4), of a measurable multifunction with convex, weakly compact values which is not the limit of any a.e. convergent sequence of simple multifunctions, convex weakly compact valued.

Although the space $(cb(X), h)$ (here h is the Hausdorff distance) is not separable, it is complete, see [4], Theorem II.14.

Definition 2.1 If $F : \Omega \to cb(X)$ is a simple measurable multifunction, namely

$$F = \sum_{i=1}^{n} C_i 1_{A_i}, \qquad C_i \in cb(X), i = 1, \cdots, n$$

we define its Bochner integral as follows: for every $E \in \Sigma$,

$$(B) - \int_E F d\mu = \sum_{i=1}^{n} C_i \mu(A_i \cap E).$$

The integral does not depend upon the representation of F and, if $F, G \in \mathcal{E}^1_{cb}(X)$, then, for every $E \in \Sigma$

$$(B) - \int_E (F \dot{+} G) d\mu = (B) - \int_E F d\mu \dot{+} (B) - \int_E G d\mu.$$

Definition 2.2 A totally measurable multifunction $F : \Omega \to cb(X)$ is *Bochner-integrable* (shortly *(B)-integrable*) iff there exists a sequence $(F_n)_n$ of simple multifunctions, $F_n \in \mathcal{E}^1_{cb}(X)$, such that

(i) $h(F_n, F)$ converges to zero μ-a.e.;

(ii) $\lim_{m,n\to\infty} \int_\Omega h(F_n, F_m) d\mu = 0$.

We shall say that $(F_n)_n$ is a *defining sequence* for F. Then, by our definition, for every measurable set $E \in \Sigma$, the sequence $\left((B) - \int_E F_n d\mu \right)_n$ is Cauchy in $(cb(X), h)$. Consequently it converges in $(cb(X), h)$, so that we can define the (B)-integral of F over E as

$$(B) - \int_E F d\mu := \lim_{n\to\infty} (B) - \int_E F_n d\mu.$$

We observe that the integral is well defined ([10]) and we denote by $L^1_{cb}(X)$ the space of all (B)-integrable multifunctions.

According to the notations adopted in [9], $L^1_{cb}(X)$ is the Δ-closure of $\mathcal{E}^1_{cb}(X)$, where Δ is the distance

$$\Delta(F, G) = \int_\Omega h(F, G) d\mu.$$

Generally we have the following relationship (see [9], (3.4)):

$$L^1_{cb}(X) \subset \mathcal{L}^1_{cb}(X)$$

and the two spaces are closed under addition $\dot{+}$ by Theorem 3.5 of [9].

Proposition 2.3 *The Bochner integral is additive, namely if* $F, G \in L^1_{cb}(X)$ *then* $F \dot{+} G \in L^1_{cb}(X)$ *and, for every* $E \in \Sigma$,

$$(B) - \int_E (F \dot{+} G) d\mu = (B) - \int_E F d\mu \dot{+} (B) - \int_E G d\mu.$$

Proof : it is obvious.

Moreover:

Theorem 2.4 *Let* F, G *be (B)-integrable multifunctions. Then*

$$\left| (B) - \int F d\mu \right| \leq \int |F| d\mu,$$

$$h\left((B) - \int F d\mu, (B) - \int G d\mu \right) \leq \int h(F, G) d\mu.$$

Proof: the proof is analogous to the one given in Theorem 2.6 of [10]

3 Comparison between the integrals

Our main interest is to give sufficient conditions for the coincidence of the Aumann and the Bochner integral for totally measurable multifunctions $F : \Omega \to cb(X)$.

Before stating our main result, Theorem 3.11, we summarize some properties.

Lemma 3.1 *If* $C \in cb(X)$ *then*

$$C = \bigcap_{\varepsilon > 0} \left(C \dot{+} \varepsilon \overline{B}_X \right).$$

Proof: Obviusly $C \subset \bigcap_{\varepsilon > 0} \left(C \dot{+} \varepsilon \overline{B}_X \right)$. Let $x \in \bigcap_{\varepsilon > 0} \left(C \dot{+} \varepsilon \overline{B}_X \right)$ and let $\varepsilon_n \downarrow 0$.

Since $x \in \bigcap_{\varepsilon_n > 0} \left(C \dot{+} \varepsilon_n \overline{B}_X \right)$, for every $n \in I\!N$ there exist $x_n \in C$ and $u_n \in \overline{B}_X$ such that $\|x - x_n - \varepsilon_n u_n\| \leq \varepsilon_n$.

$\varepsilon_n u_n$ converges strongly to zero since the sequence $(u_n)_n$ is bounded, then x_n converges to x. The conclusion follows from the closedness of C.

Proposition 3.2 *Let $F \in \mathcal{E}^1_{cb}(X)$. Then, for every $E \in \Sigma$, we have*

$$cl\{(A) - \int_E F d\mu\} = (B) - \int_E F d\mu.$$

Proof: Obviously $S^1_F \neq \emptyset$. Let $F = \sum_{i=1}^n C_i 1_{E_i},\ E_i \cap E_j = \emptyset$ for $i \neq j$; then, for every $E \in \Sigma$, we have

$$\begin{aligned} \sum_{i=1}^n C_i \mu(E \cap E_i) &= cl\{\sum_{i=1}^n x_i \mu(E \cap E_i), x_i \in C_i, i = 1, \cdots, n\} = \\ = \quad & cl\{\int_E (\sum_{i=1}^n x_i 1_{E_i}) d\mu,\ x_i \in C_i, i = 1, \cdots, n\} \subset cl\{(A) - \int_E F d\mu\} \end{aligned}$$

and so

$$(B) - \int_E F d\mu \subset cl\{(A) - \int_E F d\mu\}.$$

We are now proving the converse inclusion. Let $s \in S^1_F$. We may suppose that $s(\omega) \in F(\omega)$ for every $\omega \in \Omega$. Since $s : \Omega \to X$ is integrable, it admits a defining sequence $(s_n)_n$ of simple functions. Let $\varepsilon > 0$. Let δ be such that

(3.2.1) $$\mu(E) \leq \delta \text{ implies } \int_E |F| d\mu \leq \varepsilon.$$

By the Egoroff's Theorem, there exists $\Omega_\delta \in \Sigma$ with $\mu(\Omega - \Omega_\delta) \leq \delta$ and $(s_n)_n$ converges uniformly to s on Ω_δ. Fix $\sigma > 0$. There exists a positive integer N_σ such that for every $n \geq N_\sigma \ \ \forall\ \omega \in \Omega_\delta, s_n(\omega) \subset s(\omega) + \sigma \overline{B}_X$. Thus, for each i, and for $\omega \in \Omega_\delta \cap E_i$, we have $s_n(\omega) \in C_i \dot{+} \sigma \overline{B}_X$ whenever $n \geq N_\sigma$. Now we assert that

(3.2.2) $$\int_{E \cap \Omega_\delta} s_n d\mu \in \sum_{i=1}^n \mu(E \cap E_i \cap \Omega_\delta)(C_i \dot{+} \sigma \overline{B}_X) \subset (B) - \int_{E \cap \Omega_\delta} F d\mu \dot{+} \sigma \mu(E \cap \Omega_\delta) \overline{B}_X.$$

To simplify the reading set $\alpha_i = \mu(E \cap E_i \cap \Omega_\delta)$ and $\alpha = \mu(E \cap \Omega_\delta)$, so that $\sum_{i=1}^n \alpha_i = \alpha$ and

(3.2.3) $$\sum_{i=1}^n \alpha_i (C_i \dot{+} \sigma \overline{B}_X) \subseteq \sum_{i=1}^n \alpha_i C_i \dot{+} \sigma \alpha \overline{B}_X.$$

Hence **(3.2.2)** follows by our definition of (B)-integral for simple multifunctions. Since $\lim_{n \to \infty} \int_{E \cap \Omega_\delta} s_n d\mu = \int_{E \cap \Omega_\delta} s d\mu$, for every $\varepsilon > 0$ there exists $\overline{n} > N_\sigma$ such that, for all $n > overlinen$,

$$\left\| \int_{E \cap \Omega_\delta} s_n d\mu - \int_{E \cap \Omega_\delta} s d\mu \right\| \leq \varepsilon$$

and then, by **(3.2.2)**,

$$\begin{aligned}\int_{E\cap\Omega_\delta} s d\mu &\in \varepsilon\overline{B}_X + \int_{E\cap\Omega_\delta} s_n d\mu \\ &\subset \varepsilon\overline{B}_X \dot{+} \left((B)-\int_{E\cap\Omega_\delta} F d\mu \dot{+} \sigma\mu(E\cap\Omega_\delta)\overline{B}_X\right).\end{aligned}$$

So

$$\int_{E\cap\Omega_\delta} s d\mu \in \bigcap_{\varepsilon>0} \varepsilon\overline{B}_X \dot{+} \left((B)-\int_{E\cap\Omega_\delta} F d\mu \dot{+} \sigma\mu(E\cap\Omega_\delta)\overline{B}_X\right)$$

and, by Lemma 3.1 we obtain that

(3.2.4) $$\int_{E\cap\Omega_\delta} s d\mu \in (B)-\int_{E\cap\Omega_\delta} F d\mu \dot{+} \sigma\mu(E\cap\Omega_\delta)\overline{B}_X.$$

On the other hand, by **(3.2.1)**, we get

(3.2.5) $$\left\|\int_{E\cap\Omega_\delta^c} s d\mu\right\| \leq \int_{E\cap\Omega_\delta^c} \|s\| d\mu \leq \int_{\Omega_\delta^c} |F| d\mu \leq \varepsilon.$$

Then **(3.2.4)** and **(3.2.5)** yield

(3.2.6) $$\int_E s d\mu \in (B)-\int_{E\cap\Omega_\delta} F d\mu \dot{+} \sigma\mu(E\cap\Omega_\delta)\overline{B}_X \dot{+} \varepsilon\overline{B}_X.$$

By virtue of the classical cancellation law, we first have

(3.2.7)

$$\begin{aligned} & h\left((B)-\int_{E\cap\Omega_\delta} F d\mu \dot{+} \sigma\mu(E\cap\Omega_\delta)\overline{B}_X \dot{+} \varepsilon\overline{B}_X, (B)-\int_E F d\mu \dot{+} \sigma\mu(E)\overline{B}_X \dot{+} \varepsilon\overline{B}_X\right) \leq \\ \leq\ & h\left((B)-\int_{E\cap\Omega_\delta} F d\mu, (B)-\int_E F d\mu\right) + \sigma[\mu(E)-\mu(E\cap\Omega_\delta)] \leq \\ \leq\ & h\left((B)-\int_{E\cap\Omega_\delta^c} F d\mu, \{0\}\right) + \sigma[\mu(E-\Omega_\delta)] \leq \int_{E\cap\Omega_\delta^c} |F| d\mu + \sigma\delta \leq \varepsilon + \sigma\delta \end{aligned}$$

by the choice of σ, δ and ε. Then it follows from **(3.2.6)** and the estimate **(3.2.7)**

$$\int_E s d\mu \in (B)-\int_E F d\mu \dot{+} \left[\sigma(\mu(E)+\delta)+2\varepsilon\right]\overline{B}_X.$$

Since $(B)-\int_E F d\mu$ is non empty, bounded, closed and convex, by Lemma 3.1, we deduce that

$$\int_E s d\mu \in (B)-\int_E F d\mu,$$

and then

$$cl\{(A)-\int_E F d\mu\} \subset (B)-\int_E F d\mu.$$

Theorem 3.3 *Let $F \in \mathcal{M}^1_{cb}(X)$. If F is (B)-integrable, then for every $E \in \Sigma$, we have*

$$cl\{(A)-\int_E F d\mu\} \subset (B)-\int_E F d\mu.$$

Proof: Let $(F_n)_n$ be a defining sequence for F. Let s be an integrable selection of F; then there exists a sequence of simple functions $(s_n)_n$ which converges to s in $L^1(\Omega, \Sigma, \mu, X)$. We denote with the same simbol $(s_n)_n$ a subsequence which converges to s almost everywhere. Since $d(s(\omega), F_n(\omega)) \leq h(F(\omega), F_n(\omega)) \to 0$ and $d(s_n(\omega), F_n(\omega)) \leq \|s_n(\omega) - s(\omega)\| + d(s(\omega), F_n(\omega))$, we have that $\lim_{n\to\infty} d(s_n, F_n) = 0$ $\mu-$a.e.. We can represent s_n and F_n with the same decomposition $(E^n_j), j = 1, \cdots, p_n$ of Ω:

$$F_n = \sum_{j=1}^{p_n} C^n_j 1_{E^n_j} \qquad\qquad s_n = \sum_{j=1}^{p_n} \sigma^n_j 1_{E^n_j},$$

so that $d(s_n, F_n)$ is constant on each E^n_j; for every $n \in I\!N$ and $j \in \{1, \cdots, p_n\}$ there is $x^n_j \in C^n_j$ such that $\|\sigma^n_j - x^n_j\| \leq d(\sigma^n_j, C^n_j) + \frac{1}{n}$; thus $t_n(\omega) := \sum_{j=1}^{p_n} x^n_j 1_{E^n_j}(\omega) \in F_n(\omega)$. By construction, we have,

$$\|t_n - s\| \quad\leq\quad \|t_n - s_n\| + \|s_n - s\| \leq d(s_n, F_n) + \|s_n - s\| + \frac{1}{n}$$

so $\lim_{n\to\infty}\|t_n - s\| = 0$ $\mu-$a.e. Moreover

$$\begin{aligned}\int_E \|t_n - s\| d\mu &\leq \int_E d(s_n, F_n) d\mu + \int_E \|s_n - s\| d\mu + \frac{1}{n}\mu(E) \leq \\ &\leq \int_E h(F_n, F) d\mu + \int_E \|s_n - s\| d\mu + \frac{1}{n}\mu(E).\end{aligned}$$

Hence, for every $E \in \Sigma$, $\lim_{n\to\infty}\int_E \|t_n - s\| d\mu = 0$; then, since $\int_E t_n d\mu \in (B)-\int_E F_n d\mu$, and $\int_E s d\mu \in (A)-\int_E F d\mu$, for $\varepsilon > 0$ fixed there is a N_ε such that $n \geq N_\varepsilon$ implies

$$\int_E s d\mu \in (B)-\int_E F_n d\mu \dot{+} \varepsilon \overline{B}_X.$$

By our definition of (B)-integral, we have

$$\lim_{n\to\infty}(B)-\int_E F_n d\mu = (B)-\int_E F d\mu,$$

whence, by Lemma 3.1,

$$\int_E s d\mu \in (B)-\int_E F d\mu.$$

The converse inclusion is known when X is reflexive, see [9], Theorem 4.5 2°. In this case in fact it is possible to use the Rädstrom embedding theorem.

In order to prove the equality in the case when X is not reflexive we recall the following results:

Theorem 3.4 ([6], Theorem 5) *Assume X^* has the Radon-Nikodym property. Then every bounded sequence in $L^1(\mu, X)$ has a subsequence whose arithmethic means are weakly Cauchy almost surely.*

Theorem 3.5 ([6], Theorem 6) *Assume X^* has the Radon-Nikodym property, and let A be a bounded subset in $L^1(\mu, X)$. Then A is weakly relatively compact if and only if A is uniformly integrable and every sequence in A has a subsequence whose arthmethic means are weakly convergent almost surely.*

Theorem 3.6 ([7], Theorem 2.1)*Let A be a bounded subset of $L^1(\mu, X)$. Then the following are equivalent:*

(1) *A is weakly relatively compact;*

(2) *A is uniformly integrable, and, given any sequence $(f_n)_n$ in A, there exists a sequence $(g_n)_n$, with $g_n \in co\{f_k, k > n\}$ such that $(g_n)_n$ is norm convergent in X a.e. in Ω.*

Remark 3.7 We observe that if $F : \Omega \to cb(X)$ is integrably bounded and measurable then S^1_F is a bounded subset of $L^1(\mu, X)$, moreover it is convex because F has convex values and is closed: in fact if $(f_n)_n$ is a sequence in S^1_F such that f_n converges to f_0 in $L^1(\Omega, \Sigma, \mu, X)$ then almost everywhere f_n converge to f_0 so, for every $\varepsilon > 0$ there exists a $k(\varepsilon) \in I\!N$ such that, for every $n > k(\varepsilon)$, μ-almost everywhere $\|f(\omega) - f_n\| \leq \varepsilon$. But $f_n \in S^1_F$, so $f_0(\omega) \in F(\omega)\dot{+}\varepsilon\overline{B}_X$. Applying Lemma 3.1 $f_0(\omega) \in F(\omega)$, μ-almost everywhere.

From now on we suppose that X^* has the Radon-Nikodym property.

Lemma 3.8 *If $F : \Omega \to cb(X)$ is integrably bounded and measurable then S^1_F is weakly relatively compact.*

Dim: By Remark 3.7 S^1_F is a bounded closed convex subset of $L^1(\mu, X)$ moreover

$$\lim_{\mu(E)\to 0} \int_E f d\mu = 0,$$

uniformly for $f \in S^1_F$, in fact there exists $g \in L^1(\mu)$ such that for every $f \in S^1_F$ and for every $E \in \Sigma$,

$$\int_E |f| d\mu \leq \int_E g d\mu,$$

so the assertion follows from the absolute continuity of the integral of g with respect to μ and using Theorems 3.4 and 3.5.

As a consequence we obtain:

Proposition 3.9 *Let $F : \Omega \to cb(X)$ be a measurable, integrably bounded multifunction. Then $(A) - \int F d\mu$ is a closed subset of X.*

Proof: Let $(x_n)_n$ be a sequence in $(A) - \int F d\mu$ which converges to x_0. By definition there exists $(f_n)_n$ in S^1_F such that $x_n = \int f_n d\mu$. By Lemma 3.8, since for a subset of a Banach space the weak relative compactness is equivalent to the weak relative sequential compactness, there exists $f_0 \in S^1_F$ such that $x_0 = \int f_0 d\mu$.

Proposition 3.10 *Let $F : \Omega \to cb(X)$ be a (B)-integrable multifunction, then there exist a measurable, integrably bounded multifunction $G : \Omega \to cb(X)$ and a sequence of simple multifunctions $(G_n)_n$ such that*

(3.10.1) *$(G_n)_n$ is defining for F;*

(3.10.2) *for every $n \in I\!N$ $G_n(\omega) \subset G(\omega)$ μ-a.e.*

Proof: From Theorem 2.4 F is integrably bounded and let g be a μ-integrable scalar function which dominates F. Let $G(\omega) = 2g(\omega)\overline{B_X}$. Since g is measurable and X is separable then G admits a Castaing representation and hence G is measurable by Proposition III.9 of [4]. Since $2g$ is μ-integrable it is possible to construct an increasing sequence $(g_n)_n$ of simple functions which converges to g. Let now $(F_n)_n$ be a defining sequence for F and denote by Ω_0 the set of $\omega \in \Omega$ such that $F_n(\omega)$ does not converge to $F(\omega)$ or $|F(\omega| > g(\omega)$, by Ω_1 the set of points ω such that $g(\omega) = 0$ and by Ω_n the set of points ω such that $F_n(\omega) \cap g_n(\omega)\overline{B_X} = \emptyset$. We consider now the sequence of simple multifunctions defined as follows:

$$G_n(\omega) = \begin{cases} \{0\} & \text{if } \omega \in \Omega_0 \cup \Omega_1 \cup \Omega_n; \\ F_n(\omega) \cap g_n(\omega)\overline{B_X} & \text{otherwise.} \end{cases}$$

For every $n \in I\!N$ G_n is measurable since F_n, g_n are measurable and simple. We want to show that $(G_n)_n$ satisfies **3.10.1** and **3.10.2**. We claim that G_n converges to F μ-a.e..
For every $\omega \in \Omega_1$ $F(\omega) = G_n(\omega) = \{0\}$.
Suppose now that $\omega \in \Omega \setminus (\Omega_0 \cup \Omega_1)$. Let $\varepsilon > 0$ be such that $2g(\omega) - \varepsilon > g(\omega) + \varepsilon$. Then there exists $\overline{n} \in I\!N$ such that, for every $n \geq \overline{n}$,

$$g(\omega) + \varepsilon < 2g(\omega) - \varepsilon \leq g_n(\omega); \qquad h(F_n(\omega), F(\omega)) \leq \varepsilon.$$

So for every $n \geq \overline{n}$

$$F_n(\omega) \subset F(\omega) \dot{+} \varepsilon \overline{B}_X \subset (g(\omega) + \varepsilon)\, \overline{B}_X$$

and then $\omega \notin \Omega_n$, and $F_n(\omega) \subset g_n(\omega)\overline{B}_X$, so $G_n(\omega) = F_n(\omega)$ and converges to F in $\Omega \setminus \Omega_0$. Moreover, since $h(G_n(\omega), \{0\}) \leq 2g(\omega)$ for every $\omega \in \Omega$, $\int h(G, \{0\})d\mu$ is absolutely continuous with respect to μ uniformly in $n \in I\!N$ and hence by Vitali Theorem 2.14, of [10], for every $E \in \Sigma$,

$$\lim_{n\to\infty} \int_E G_n d\mu = \int_E F d\mu.$$

We are now ready to prove the following equality:

Theorem 3.11 *Let $F : \Omega \to cb(X)$ be a (B)-integrable multifunction. Then, for every $E \in \Sigma$*

$$(A) - \int_E F d\mu = (B) - \int_E F d\mu.$$

Proof: One inclusion is given in Theorem 3.3; we now prove the converse inclusion, that is

$$cl\{(A) - \int_E F d\mu\} = (A) - \int_E F d\mu \subset (B) - \int_E F d\mu.$$

Let G and $(G_n)_n$ be defined as in Proposition 3.10. For every $E \in \Sigma$ fixed let $t_E \in (B) - \int_E F d\mu$.

$$\begin{aligned} d\left(t_E, (B) - \int_E G_n d\mu\right) &\leq h\left((B) - \int_E G_n d\mu, \int_E F d\mu\right) \leq \\ &\leq \int_E h(G_n, F) d\mu \leq \int_\Omega h(G_n, F) d\mu, \end{aligned}$$

so, since for every $n \in I\!N$, $(B) - \int_E G_n d\mu = (A) - \int_E G_n d\mu$, there exists a sequence $(f_n)_n \in S^1_{G_n}$ such that, $\int_E f_n d\mu \to t_E$. Since for every $n \in I\!N$, $S^1_{G_n} \subset S^1_G$, it follows that $\bigcup_n S^1_{G_n} \subset S^1_G$. G satisfies all the hypothesis of Lemma 3.8 so S^1_G is weakly sequentially compact. Then there exists $f_0 \in S^1_G$ and a subsequence $(f_{n_k})_{n_k}$ of $(f_n)_n$ such that, for every $A \in \Sigma$,

$$\lim_{n_k \to \infty} \int_A f_{n_k} d\mu = \int_A f_0 d\mu.$$

So $t_E = \int_E f_0 d\mu$. We have now to proof that Let $(g_k)_k = (f_{n_k})_{n_k}$ be the subsequence which converges weakly to f_0. Since $(G_{n_k})_{n_k}$ is defining for F, there exists a set N with $\mu(N) = 0$ such that for every $\omega \notin N$ and for every $\varepsilon > 0$ there exists $n(\varepsilon, \omega)$ such that for every $n_k \geq n(\varepsilon)$,

$$d(g_k, F) \leq h(G_{n_k}, F) \leq \varepsilon.$$

From (2) of Theorem 3.6 there exists some sequence of convex combination of g_k which converges to f_0 in X-norm. Let $l = \sum_{j=1}^{p} \alpha_j g_{k_i}$, $\sum_{j=1}^{p} \alpha_j = 1$ be a convex combinations of $(g_k)_k$ such that

$$\|l(\omega) - f_0(\omega)\| \leq \varepsilon.$$

Since F is convex valued

$$\begin{aligned} d(l(\omega), F(\omega)) &= d(\sum_{j=1}^{p} \alpha_j g_{k_i}(\omega), F(\omega)) \leq \sum_{j=1}^{p} \alpha_j d(g_{k_i}(\omega), F(\omega)) \leq \\ &\leq \sum_{j=1}^{p} \alpha_j h(G_{n_{k_i}}(\omega), F(\omega)) \leq \varepsilon. \end{aligned}$$

So,

$$d(f_0, F) \leq \|f_0 - l\| + d(l, F) \leq 2\varepsilon.$$

For the arbitrariness of $\varepsilon > 0$ we obtain that $d(f_0, F) = 0$ μ-almost everywhere, hence $f_0(\omega) \in F(\omega)$ μ-almost everywhere.

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