A Radon-Nikodym theorem for the Bartle-Dunford-Schwartz integral with respect to finitely additive measures

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1. Introduction

One the most recent development in vector integration is directed toward defining the integral in a locally convex space. This generalization is not artificial, but follows the current investigation concerning, for example, Stochastic Processes. The existence of a density is indeed a fundamental tool for the decomposition theorems that allow to single out the "good" integrators in the theory of Stochastic Integration. The


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setting of locally convex spaces thus makes it possible to develop a theory of Stochastic integration e.g. in nuclear spaces as the space of distributions ([2]).

The integral of a scalar function with respect to a vector finitely additive measure $\mu$ can be defined in several different ways (see [7]). In particular, one can consider the Bartle–Dunford–Schwartz integral defined in the following way: let $X$ be a locally convex topological vector space, $(\Omega, \Sigma)$ a measurable space, $\mu : \Sigma \rightarrow X$ a finitely additive measure, $f$ a scalar valued function such that for every $x^* \in X^*$, $f \in L^1(x^* \mu)$; then $f$ is \textit{integrable in the Bartle–Dunford–Schwartz sense} if for every $E \in \Sigma$ there exists $x_E \in X$ such that

$$x^*(x_E) = \int_E f dx^* \mu$$

for every $x^* \in X^*$. For this integral Musia[10] has given a Radon-Nikodym Theorem when $\mu$ is countably additive: he obtains the equivalence of the existence of a density with three equivalent conditions expressing the suitable \textit{absolute continuity}.

The aim of this paper is to extend the Radon-Nikodym Theorem of [10] to the case of finitely additive measure’s.

The Radon-Nikodym Theorem here proven makes use of the Moedomo-Uhl kind of assumption [9].

The first complication arises from the fact that the finite additivity, due to the lack of the Radon-Nikodym Theorem even in the scalar setting, does not guarantee under the simple assumption of the absolute continuity the existence of the scalar density $\frac{dx^* \nu}{dx^* \mu}$. Hence one has to assume such existence or some conditions ensuring
it, like those in [4]. Moreover, since the proof in the countably additive case is based upon the existence of a lifting, it cannot be mimicked in the present setting; its role in the proof is somehow replaced by assuming that $\mu$ admits a Rybakov control. This condition, which is not necessarily satisfied when $X$ is a locally convex topological vector space even for $s$-bounded $\mu$ (as it is when $X$ is a Banach space), is shortly discussed at the end of the paper.

2. Preliminaries

Throughout the sequel $X$ will be a sequentially complete locally convex topological vector space. Let $(\Omega, \Sigma)$ be a measurable space.

(C1) Let $\nu, \mu : \Sigma \to X$ be two finitely additive measures (f.a.m.'s) such that for every $x^* \in X^*$ the f.a.m.'s $x^*\mu$ and $x^*\nu$ are b.v. Assume also that $\mu$ admits a Rybakov control $\lambda = |x_0^*\mu|$. We begin with some definitions.

DEFINITION 1. We shall say that $\nu$ is scalarly uniformly absolutely continuous with respect to $\mu$, and write $\nu \ll \mu$ if for every $\varepsilon > 0$ there exists $\delta > 0$ such that for every $x^* \in X^*$ and $E \in \Sigma$ $|x^*\mu|(E) < \delta$ yields $|x^*\nu|(E) < \varepsilon$.

DEFINITION 2. We shall say that $\nu$ is scalarly dominated by $\mu$ if there exists $M > 0$ such that $|x^*\nu|(E) \leq M|x^*\mu|(E)$ for every $E \in \Sigma$ and $x^* \in X^*$.
DEFINITION 3. We shall say that $\nu$ is *subordinated to* $\mu$ if there exists $N > 0$ such that for every $E \in \Sigma$ $\nu(E) \in \text{aco}\{\mu(F), F \in E \cap \Sigma\}$ where $\text{aco}(A) = \{\sum_{i=1}^{n} \alpha_i x_i, x_i \in A, \sum_{i=1}^{n} |\alpha_i| = 1\}$.

Let $\mathcal{P}$ be the family of seminorms generating the topology of $\mathcal{X}$.

We shall say that the range of $\mu$ $R(\mu)$ is *bounded* if for every $p \in \mathcal{P}$ there exists $\lambda_p > 0$ such that $R(\mu) \subset \lambda_p\{x \in \mathcal{X} : p(x) \leq 1\}$.

If $R(\mu)$ is bounded, we shall set

$$\mathcal{P}_{\mu} = \{\frac{p}{\lambda_p}, p \in \mathcal{P}\}$$

and

$$\mathcal{X}^*_{\mu,p} = \{x^* \in \mathcal{X}^* : x^* \leq p\}.$$

Let

$$G_{1,\mu} = \{f : \Omega \to \mathbb{R} : f \in L^1(|x^*\mu|) \forall x^* \in \bigcup_{p \in \mathcal{P}_{\mu}} \mathcal{X}^*_{\mu,p}\}.$$

DEFINITION 4. Let $f \in G_{1,\mu}$; we shall say that $f$ is *$\mu$-integrable* provided for every $A \in \Sigma$ there exists $v(A) \in \mathcal{X}$ such that

$$x^*v(A) = \int_A fd(x^*\mu)$$

for every $x^* \in \mathcal{X}^*$. Then we shall set $\int_A fd\mu = v(A)$.

We shall need in the sequel the following finitely additive extension of a classical theorem

THEOREM 1. *(Image Law)* Let $m, s : \Sigma \to \mathbb{R}$ be two f.a.m.'s with bounded variation. If $s = \int f dm$ and $h : \Omega \to \mathbb{R}$ is a $\Sigma$-measurable bounded function, then $fh \in L^1(|m|)$ and

$$\int hds = \int hf dm.$$

(1)
Proof. As \( h \) is bounded, there exists \( M > 0 \) such that \( |h(\omega)| \leq M \) for every \( \omega \in \Omega \) and thus \( |hf| \leq M |f| \in L^1(|m|) \): hence the \( m \)-integrability of \( hf \) is straightforward. It remains to prove the equality in (1).

Let \( h \) be simple: then (1) is obvious. Assume now that \( h \geq 0 \); then the Lebesgue ladder trick gives a sequence \((h_n)_n\) of simple functions such that

\[
h_n \leq h_{n+1} \leq h \quad \text{for every} \quad n \in \mathbb{N};
\]

\( h_n \) converges uniformly to \( h \).

By the \( m \)-integrability of \( f \) there exists a defining sequence of simple functions \((f_n)_n\) such that \( f_n \rightarrow f \) in \( L^1(|m|) \) ([7]). Let \( s_n(\cdot) = \int f_n \, dm \); then for every \( \varepsilon > 0 \) there exists \( n_0 \in \mathbb{N} \) such that for all \( n > n_0 \)

\[
|s_n - s|(E) = \int_E |f_n - f| \, dm \leq \int_\Omega |f_n - f| \, dm < \varepsilon
\]

for every \( E \in \Sigma \), namely \( |s_n - s|(\cdot) \) converges to 0 uniformly in \( \Sigma \).

Let \( E \in \Sigma \) be fixed and let us put \( a_{i,n}(E) = \int_E h_n \, ds_i \); then, since \( h_n \) and \( f_i \) are simple, \( a_{i,n}(E) = \int_E h_n f_i \, dm \). We shall show that for every \( i \in \mathbb{N} \) there exists \( \lim_{n \to \infty} a_{i,n}(E) \) and that \( \lim_{i \to \infty} a_{i,n}(E) \) exists uniformly in \( n \in \mathbb{N} \). Then it will follow that

\[
\lim_{i \to \infty} \lim_{n \to \infty} a_{i,n}(E) = \lim_{n \to \infty} \lim_{i \to \infty} a_{i,n}(E) = \lim_{i,n \to \infty} a_{i,n}(E).
\]

Let \( i \in \mathbb{N} \) be fixed; from the uniform convergence of \( h_n \) to \( h \) we have that \( h_n \) \( s_i \)-converges to \( h \) and \( \int_E h_n \, ds_i \rightarrow \int_E h \, ds_i \) that is \( \lim_{n \to \infty} a_{i,n}(E) \) exists.

Moreover, if \( n \) is fixed, as \( h_n \) is simple, it is easy to check that

\[
|a_{i,n}(E) - \int_E h_n \, ds| = \left| \int_E h_n \, ds_i - \int_E h_n \, ds \right| \leq M |s_i - s|(E)
\]
and since, as observed, $|s_i - s|(\Omega) \to 0$ it follows that $\lim_{i \to \infty} a_{i,n}(E) = \int_E h_n ds$ uniformly with respect to $n$.

Then

$$\int_E hf dm = \lim_{i,n \to \infty} \int_E h_n f dm = \int_E h ds.$$  

For general bounded $h$ it is enough to decompose $h = h^+ - h^-$.

**DEFINITION 5.** We shall say that a measurable function $f : \Omega \to \mathbb{R}$ is $\lambda$-null if for every $\varepsilon > 0$ it is $\lambda(\{|f| > \varepsilon\}) = 0$.

Observe that if $f = 0$ $\lambda$-a.e. then $f$ is $\lambda$-null, while the converse is true if $\lambda$ is $\sigma$-additive, or at least $\lambda$ fulfills the condition

$(\sigma)$ the ideal of $\lambda$-null sets is closed under countable unions.

We list two straightforward properties of $\lambda$-null functions that we will need in the sequel:

**(p.1)** $f$ is $\lambda$-null iff $\int_E |f| d\lambda = 0$ for every $E \in \Sigma$;

**(p.2)** if $f$ is $\lambda$-null and $g$ is bounded then $fg$ is $\lambda$-null.

### 3. Radon-Nikodym Theorem

We shall now prove the main theorem.

**THEOREM 2. (Radon-Nikodym)** Let $(\Omega, \Sigma)$ be a measurable space, and $\nu, \mu : \Sigma \to \mathcal{X}$ be two f.a.m.’s satisfying (C1).

Assume that for every $* \in X^*$ there exists a $\lambda$-exhaustion $(\Omega_n^*)$ such that for every $n \in \mathbb{N}$ the set

$$S_n^* = \left\{ \frac{x^* \mu(A)}{\lambda(A)} : A \in \Omega_n^* \cap \Sigma, \lambda(A) > 0 \right\}$$
is bounded for every \( x^* \in \mathcal{X}^* \), and for every \( n \in \mathbb{N} \) the ranges \( R(\mu|_{\Omega_n^* \cap \Sigma}) \) and \( R(\nu|_{\Omega_n^* \cap \Sigma}) \) are closed.

Then the following are equivalent:

i) \( \nu \ll \mu \);

ii) \( \nu \) is scalarly dominated by \( \mu \);

iii) there exists \( \vartheta : \Omega \rightarrow \mathbb{R} \) bounded and such that

\[
x^* \nu(E) = \int_E \vartheta \, dx^* \mu
\]

for every \( E \in \Sigma \) and for every \( x^* \in \mathcal{X}^* \);

iv) \( \nu \) is subordinated to \( \mu \).

**Proof.** The proofs of the implications i) \( \iff \) ii) and iv) \( \implies \) ii) are essentially the same as in [6]; also the implication iii) \( \implies \) iv) can be proven in the same fashion as in the same paper, by making use of the results in [7] that are the extensions in the finitely additive setting of those of Lewis [6]. Hence it only remains to prove the implication ii) \( \implies \) iii).

**Claim 1** - For every \( x^* \in \mathcal{X}^* \) there exist \( \frac{dx^* \mu_n}{d\lambda}, \frac{dx^* \nu_n}{d\lambda} \), where \( \mu_n = \mu|_{\Omega_n^* \cap \Sigma} \) and \( \nu_n = \nu|_{\Omega_n^* \cap \Sigma} \).

**Proof.** We shall show the assertion for \( (x^* \mu, \lambda) \) since the proof for \( (x^* \nu, \lambda) \) is analogous.

Let \( x^* \in \mathcal{X}^* \) be fixed and let \( n \in \mathbb{N} \). For every \( a, b \in \mathbb{R} \) let \( y^* = ax^* + bx_0^* \); since our assumptions ensure that \( R(y^* \mu_n) \) is closed, the signed measure \( y^* \mu_n \) admits a Hahn decomposition. Then from [3] Lemma 4.3 there
exists \( \frac{dx^* \mu_n}{d|x_0^n\mu|} = \frac{dx^* \mu_n}{d\lambda} \).

We shall denote \( f^{(n)}_{x^*} = \frac{dx^* \nu_n}{d\lambda} \) and \( g^{(n)}_{x^*} = \frac{dx^* \mu_n}{d\lambda} \).

Observe also that, without loss of generality, one can choose a representative of \( g^{(n)}_{x^*} \) such that the set \( \{ g^{(n)}_{x^*} = 0 \} \) is empty, and \( |g^{(n)}_{x^*}| = 1 \).

**Claim 2 -** \( g^{(n)}_{x^*} \) is essentially bounded on \( \Omega^{x^*}_n \) for every \( n \in \mathbb{N} \).

This follows immediately from the boundedness of \( S^{x^*}_n \).

Then define \( L^{(n)}_{x^*} = \text{supess} |g^{(n)}_{x^*}| \) and let \( \Omega^{(n)}_{0,x^*} \subset \Omega^{x^*}_n \) be such that \( \lambda(\Omega^{(n)}_{0,x^*}) = 0 \) and

\[
|g^{(n)}_{x^*}(\omega)| \leq L^{(n)}_{x^*}
\]

for every \( \omega \not\in \Omega^{(n)}_{0,x^*} \). Since \( (\Omega^{x^*}_n)_n \) is a \( \lambda \)-exhaustion, \( \lambda(\bigcup_n \Omega^{(n)}_{0,x^*}) = 0 \).

Define \( \Omega'_{n,x^*} = \Omega^{x^*}_n - \Omega^{(n)}_{0,x^*} \); then \( \lambda(\bigcup_n \Omega'_{n,x^*}) = \lambda(\Omega) \) and \( (\Omega'_{n,x^*})_n \) is a \( \lambda \)-exhaustion of \( \Omega \). Since the \( \Omega'_{n,x^*} \)'s are pairwise disjoint, also the \( \Omega'_{n,x^*} \)'s are pairwise disjoint, so we can define \( g_{x^*} = \sum_{n=1}^{\infty} g^{(n)}_{x^*} 1_{\Omega'_{n,x^*}} \) and

\[
f_{x^*} = \sum_{n=1}^{\infty} f^{(n)}_{x^*} 1_{\Omega'_{n,x^*}}.
\]

Let \( x^* \in \mathcal{X}^* \) be fixed and define \( H_{x^*} = \{ g_{x^*} \neq 0 \} \).

**Claim 3 -** The function \( \frac{f_{x^*}}{g^{(n)}_{x^*} - g_{x^*}} \) is \( \lambda \)-null in \( H_{x^*} \cap \Omega'_{n,x^*} \).

**Proof.** From the linearity of the maps \( x^* \to f_{x^*} \) and \( x^* \to g_{x^*} \), and from assumption \( \text{ii) for every } E \in \Sigma \) and for every \( \beta_1, \beta_2 \in \mathbb{R} \)

\[
\int_E |\beta_1 f_{x^*} + \beta_2 g_{x^*}| d\lambda \leq M \int_E |\beta_1 f_{x^*} + \beta_2 g_{x^*}| d\lambda
\]

(2) Let \( n \) be fixed and let \( \tilde{\Omega}_{n,x^*} = \Omega'_{n,x^*} \cap H_{x^*} \). Then \( g_{x^*} \) is bounded in \( \tilde{\Omega}_{n,x^*} \) and therefore there exists a sequence of simple functions \( (\tilde{\gamma}^{(n)}_{x^*})_k \)
that converges uniformly to $g_x^{(n)}$ in $\tilde{\Omega}_{n,x^*}$; also, since in $\tilde{\Omega}_{n,x^*}$ $g_{x^*} \neq 0$ for $k$ large enough $\gamma^{(n)}_{x^*,k} \neq 0$ in $\tilde{\Omega}_{n,x^*}$.

Since $\gamma^{(n)}_{x^*,k}, g_{x_0^*}$ are simple functions, it is possible to decompose $\tilde{\Omega}_{n,x^*}$ into finitely many subsets where both these functions are constant. Let $(E^{(j)}_n)_{j=1}^{r(x^*,k)}$ be such a decomposition. Taking $\beta_1 = \frac{1}{\gamma^{(n)}_{x^*,k}(E^{(j)}_n)}$ and $\beta_2 = -\frac{1}{g_{x_0^*}(E^{(j)}_n)}$ in (2) we then find

$$\int_{E^{(j)}_n \cap E^{(j)}_n} \frac{f_x^*}{\gamma^{(n)}_{x^*,k}} - \frac{f_{x_0^*}}{g_{x_0^*}} d\lambda \leq M \int_{E^{(j)}_n \cap E^{(j)}_n} \frac{g_{x^*}}{\gamma^{(n)}_{x^*,k}} - 1 d\lambda$$

for every $j$ and since the $E^{(j)}_n$’s are finitely many, for every $E \subset \tilde{\Omega}_{n,x^*}$

$$\int_E \frac{f_x^*}{\gamma^{(n)}_{x^*,k}} - \frac{f_{x_0^*}}{g_{x_0^*}} d\lambda \leq M \int_E \frac{g_{x^*}}{\gamma^{(n)}_{x^*,k}} - 1 d\lambda$$

(3)

By taking the limit for $k \to \infty$ in (3) we then obtain

$$\int_E \frac{f_x^*}{g_{x^*}} - \frac{f_{x_0^*}}{g_{x_0^*}} d\lambda = 0$$

for every $E \subset \tilde{\Omega}_{n,x^*}$. Then (p1) implies Claim 3.

Let $\vartheta = \frac{f_{x_0^*}}{g_{x_0^*}}$. From (p. 2) the function $f_x^* - \vartheta g_{x^*}$ is $\lambda$-null in $\tilde{\Omega}_{n,x^*}$.

Claim 4 - For every $\varepsilon > 0$ $|f_x^*| \leq (M + \varepsilon)|g_{x^*}|$ $\lambda$-a.e. in $\Omega_{n,x^*}$, where $M$ is that of the scalar domination.

Proof. Indeed from the assumption ii) for every $E \in \Omega_{n,x^*} \cap \Sigma$

$$\int_E |f_x^*| d\lambda \leq M \int_E |g_{x^*}| d\lambda.$$

Hence one can easily prove that for every $\varepsilon > 0$

$$|f_x^*| < (M + \varepsilon)|g_{x^*}|$$
\( \lambda \)-a.e. in \( \Omega'_n^x \).

From Claim 4 it follows that if \( \omega \in \Omega'_n^x - H^x \), namely if \( g^x_\omega = 0 \) then \( \lambda \)-a.s. \( f^x_\omega = 0 \); therefore \( f^x = 0 \) \( \lambda \)-a.e. in \( \Omega'_n^x - H^x \) and then \( f^x - \vartheta g^x = 0 \) \( \lambda \)-a.e. in \( \Omega'_n^x - H^x \).

We can therefore conclude that \( f^x - \vartheta g^x \) is \( \lambda \)-null in \( \Omega'_n^x \).

**Claim 5** - For every \( x^* \in X^* \), \( x^* \nu(E) = \int_E \vartheta dx^* \mu \).

**Proof.** From (p.1) we find
\[
| \int_E f^x d\lambda - \int_E \vartheta g^x d\lambda | \leq \int_E | f^x - \vartheta g^x | d\lambda = 0
\]
for every \( E \subset \Omega'_n^x \) and thus for every \( E \subset \Omega'_n^x \)
\[
\int_E f^x d\lambda = \int_E \vartheta g^x d\lambda.
\]

Observe also that, since \( |g^x_0| = 1 \) from Claim 4 \( |f^x_0| < M + \varepsilon \) \( \lambda \)-a.e. in each \( \Omega'_n^x \) whence \( \vartheta \) is \( \lambda \)-a.e. bounded. From Theorem 1, for every \( E \in \Sigma \) and for \( x^* \in X^* \) fixed
\[
x^* \nu(E \cap \Omega_n^{x^*}) = \int_{E \cap \Omega_n^{x^*}} f^x d\lambda = \int_{E \cap \Omega_n^{x^*}} \vartheta g^x d\lambda = \int_{E \cap \Omega_n^{x^*}} \vartheta dx^* \mu.
\]

Since \( (\Omega_n^{x^*})_n \) is a \( \lambda \)-exhaustion of \( \Omega \), for every \( E \in \Sigma \)
\[
x^* \nu(E) = \sum_{i=1}^k x^* \nu(E \cap \Omega_i^{x^*}) + x^* \nu[E \cap ( \bigcup_{i=k+1}^{\infty} \Omega_i^{x^*})]
\]
\[
= \sum_{i=1}^k \int_{E \cap \Omega_i^{x^*}} \vartheta dx^* \mu + x^* \nu[E \cap ( \bigcup_{i=k+1}^{\infty} \Omega_i^{x^*})],
\]
while
\[
\int_E \vartheta dx^* \mu = \int_{E \cap (\bigcup_{i=1}^k \Omega_i^{x^*})} \vartheta dx^* \mu + \int_{E \cap (\bigcup_{i=k+1}^{\infty} \Omega_i^{x^*})} \vartheta dx^* \mu.
\]
whence

\[ |x^\ast \nu(E) - \int_E \vartheta \, dx^\ast \mu| \leq |\int_E \vartheta (E \cap \bigcup_{i=k+1}^{\infty} \Omega_i^+) \, dx^\ast \mu| + |x^\ast \nu(E \cap \bigcup_{i=k+1}^{\infty} \Omega_i^+)\|. \]

Since \( \lim_{k \to \infty} \lambda[E \cap \bigcup_{i=k+1}^{\infty} \Omega_i^+] = 0 \) and \( |x^\ast \nu| \ll \lambda \) and \( |x^\ast \mu| \ll \lambda \), it follows that

\[ x^\ast \nu(E) = \int_E \vartheta \, dx^\ast \mu \]

for every \( E \subset \Omega \).

**REMARK 1.**

1. In [5] Drewnovski studied the existence of a Rybakov control for an \( \mathcal{X} \)-valued countably additive measure. He showed that in general a Rybakov control does not exist in l.c.t.v. spaces unless some further conditions are satisfied. He also gave a quite strong condition for a f.a.m. to admit a Rybakov control.

2. It is easy to mimick the previous proof provided \( \mu \) admits a control \( \lambda \) such that for some \( x^\ast \in \mathcal{X}^\ast \lambda(\{ \frac{dx^\ast \mu}{d\lambda} = 0 \}) = 0 \). It could therefore be of interest to investigate whether a f.a.m. \( \mu \) admitting a control \( \lambda \) always fulfills this condition.

**References**


2. J. Brook unpublished manuscript.


