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MULTIVALUED INTEGRAL OF NON CONVEX INTEGRANDS

Anna Martellotti[1], **Anna Rita Sambucini**[2]

[1] Dept. of Mathematics and Inform. 1, Via Vanvitelli - 06123 Perugia, Italy
e-mail: amart@dipmat.unipg.it

[2] Dept. of Mathematics and Inform. 1, Via Vanvitelli - 06123 Perugia, Italy
e-mail: matears1@unipg.it

Abstract: We study here the convexity of the Aumann integral for suitable multifunctions with values in the closed subsets of an infinite dimensional spaces.

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1. Introduction

In the study of atomless economies an important role is played by the convexity and the closure of the Aumann integral of a multifunction of the form

$$F(\omega) = (\Gamma(\omega) - e(\omega)) \cup \{0\} \tag{1}$$

where e is an integrable vector function, Γ a suitable multifunction with closed and convex values.

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The Liapounov property of the Aumann integral is well known in the finitely dimensional model, see for example Hildebrand [8]; but in the infinite dimensional case the Aumann integral may lack both these properties. An example is due to A. Rustichini and N. Yannelis who take an l_2-valued multifunction of the type $F(t) = \{0, u(t)\},\ t \in [0, 2\pi]$ where u is that of Diestel,Uhl [6, Example IX.2].

In this paper we shall consider a suitable class of integrable multifunctions of type (1), which will turn out to have convex Aumann integral. This will be done in the countably additive case; in Martellotti and Sambucini [12] the finitely additive case has been considered.

X will be a reflexive separable Banach space, and the class that we consider is that of multifunctions of the type (1) where $\Gamma = \sum_{i=1}^{p} C_i 1_{E_i}$ is an X-valued simple multifunction with closed and convex values and e is a Bochner integrable function which admits a Liapounov indefinite integral.

The idea of dividing the space of traders Ω into a finite decomposition $(E_1, \ldots, E_p)$ appears for instance in Basile and Graziano [1]. There the authors give the following motivation: *"an istitutional coalition structure is imposed to the society in the form of restricted set of coalitions: the only admissible coalitions are those belonging to the given structure"*, the motivation is that *"in the real economic activity the lack of communication and information among traders and the cost of transactions restrict the set of coalitions that are going to form"*.

The kind of economic application that we have in mind is that of a "simplified economic model": namely the market Ω is divided into a finite decomposition $(E_1, \ldots, E_p)$ and the traders in each E_i share, *indipendently of their welfare,* the same preferences. This has a very clear economic interpretation.

2. Preliminaries and definitions

Let Ω be a set, Σ a σ-algebra of subsets of Ω and $\mu : \Sigma \to [0, +\infty[$ a bounded non atomic measure. Let X be a reflexive, separable Banach space. With X^* we denote its topological dual and with X_1, X_1^* the unit balls of X and X^*

respectively. We denote by X_w the space X equipped with its weak topology. We denote by $L^1_\mu(X)$ the space of Bochner integrable functions f. When $X = I\!R$ we shall simply write L^1_μ.

Definition 2.1 A vector measure $m : \Sigma \to X$ is called a *Liapounov measure* if, for every $E \in \Sigma$, $m(\Sigma_E) = \{m(A), A \in \Sigma \cap E\}$ is convex and weakly compact for every $E \in \Sigma$. Since we have assumed that X is a reflexive Banach space it is enough to assume that $m(\Sigma_E)$ is closed and convex for every $E \in \Sigma$.

If, for every $E \in \Sigma$, $m(\Sigma_E)$ is only convex, we will say that m is a *convex measure.*
We shall denote by $cf(X)$ the family of non empty, convex, closed subsets of X and by $cwk(X)$ the family of non empty, convex, weakly compact subsets of X.
A multifunction $F : \Omega \to 2^X \setminus \{\emptyset\}$ is said to be *Effros measurable* (*measurable*) if for every closed subset of X, C

$$F^-(C) = \{\omega \in \Omega : F(\omega) \cap C \neq \emptyset\} \in \Sigma.$$

A multifunction $F : \Omega \to 2^X \setminus \{\emptyset\}$ is said to be *integrably bounded* if there exists $g \in L^1_\mu$ such that, for almost every $\omega \in \Omega$

$$\|x\| \leq g(\omega), \quad \text{for every} \quad x \in F(\omega).$$

We denote by S^1_F the set of all Bochner integrable selections of F, namely

$$S^1_F = \{f \in L^1_\mu(X) : f(\omega) \in F(\omega) \quad \mu - \text{almost everywhere}\}.$$

If F is a measurable multifunction, and $S^1_F \neq \emptyset$, then the *Aumann integral* (shortly (A)-integral) of F is given by

$$(A) - \int F d\mu = \left\{ \int f d\mu, \quad \text{for every } f \in S^1_F \right\}.$$

Definition 2.2 A map $M : \Sigma \to 2^X \setminus \{\emptyset\}$ is called a *multimeasure* if $M(\emptyset) = \{0\}$ and for every sequence of disjoint sets $E_i \in \Sigma$ with $E = \bigcup_i E_i$,

$$M(E) := \sum_{i=1}^{\infty} M(E_i) = \{x \in X : x = \sum_{i=1}^{\infty} x_i,\ x_i \in M(E_i)\}.$$

Given a multimeasure $M : \Sigma \to 2^X \setminus \{\emptyset\}$, a vector measure $m : \Sigma \to X$ such that $m(E) \in M(E)$ for every $E \in \Sigma$ is called a *measure selection* of M. The set of all measure selections of M is denoted by S_M. M is called *perfect* if $M(E) = \{m(E), m \in S_M\}$.
Let M be a multimeasure and $\mathcal{H}$ be a family of measure selections of M: we shall say that $\mathcal{H}$ *fills out* M if $M(E) = \{m(E), m \in \mathcal{H}\}$ for every $E \in \Sigma$.

If we consider a multimeasure $M : \Sigma \to cwk(X)$ we shall consider the following ranges $R(M) = \{M(E), E \in \Sigma\}$, which is the range in the hyperspace $(cwk(X), h)$, and $R_X(M) = \cup_{E\in\Sigma} M(E)$, which is the range in X.

Throughout this paper we will assume always that:

- $\Gamma = \sum_{i=1}^{p} C_i 1_{E_i}$ is a simple multifunction with values in $cf(X)$ where $(E_1, \cdots, E_p)$ is a finite decomposition of Ω (namely the $E_i's$ are pairwise disjoint and $\bigcup_{i=1}^{p} E_i = \Omega$);
- $e \in L^1_\mu(X)$ is such that the measure $\lambda(E) := -\int_E e d\mu$ is Liapounov;
- $$F = G \cup \{0\} = (\Gamma - e) \cup \{0\}.$$

3. Properties of the Aumann integral in the countably additive setting

We shall first assume that $0 \notin G(\omega)$, for every $\omega \in \Omega$. This last assumption does not restrict the generality of the problem, as we will see in subsection 3.3.

3.1. Integrands with bounded values

We shall begin considering multifunctions with bounded values; in other words we assume that $C_i \in cwk(X)$ $i = 1, \ldots, p$. First of all we want to prove that the Aumann integral of $G = \Gamma - e$ is convex and weakly compact. In fact, in general,

Proposition 3.3 *If $\Phi : \Omega \to cwk(X)$ is a totally measurable integrably bounded multifunction then, for every $E \in \Sigma$,*

$$(A) - \int_E (\Phi - e) d\mu = (A) - \int_E \Phi d\mu - \int_E e d\mu \in cwk(X).$$

Proof: it is an easy consequence of the definition and of Byrne's result [2].

Given a multifunction Ψ, we shall denote with $M_\Psi : \Sigma \to 2^X$ the map defined by:

$$M_\Psi(E) = (A) - \int_E \Psi d\mu.$$

We prove now that:

Proposition 3.4 *If $\Gamma : \Omega \to cwk(X)$ is simple M_Γ is a multimeasure and*

$$M_\Gamma(E) := \{m_f(E) := \int_E f d\mu, \quad f \in S^1_\Gamma, f \text{ simple } \}.$$

Proof: We remember that, since Γ is simple, namely $\Gamma(\omega) = \sum_{i=1}^p C_i 1_{E_i}(\omega)$, by Byrne [2], then it is Aumann and Debreu integrable and

$$(A) - \int_E \Gamma d\mu = (D) - \int_E \Gamma d\mu = \sum_{i=1}^p C_i \mu(E \cap E_i). \qquad (2)$$

Then, if $x \in (A) - \int_E \Gamma d\mu$ there exist $x_i \in C_i,\ i = 1, \ldots, p$ such that $x = \sum_{i=1}^p x_i \mu(E \cap E_i)$. But then, setting $f = \sum_{i=1}^p x_i 1_{E_i}$, it is clear that $f \in S^1_\Gamma$ and $x = m_f(E)$. Therefore

$$M_\Gamma(E) \subset \{m_f(E) = \int_E f d\mu, \quad f \in S^1_\Gamma, f \text{ simple } \}.$$

The converse inclusion is obvious. Moreover, via Radström embedding theorem ([13]), since the Debreu integral is countably additive, if $(A_n)_n$ is a disjoint sequence of Σ-measurable sets and we denote by A its union then

$$M_\Gamma(A) = (D) - \int_A \Gamma d\mu = \sum_{n=1}^\infty (D) - \int_{A_n} \Gamma d\mu = \sum_{n=1}^\infty M_\Gamma(A_n). \ \square$$

Remark 3.5 Let $\mathcal{H}$ be the family

$$\mathcal{H} = \{m_f \in S_{M_\Gamma} : \quad f \in S^1_\Gamma, f = \sum_{i=1}^p x_i 1_{E_i},\ x_i \in C_i\}. \qquad (3)$$

Proposition 3.4 says then that $\mathcal{H}$ fills out M_Γ.

We prove now the convexity and the closure of the range in X of the multimeasure M_Γ.

Proposition 3.6 *$R_X(M_\Gamma)$ is convex and weakly compact.*

Proof: Indeed we shall prove that $R_X(M_\Gamma) = \sum_{i=1}^{p} co\,(\{0\} \cup C_i)\,\mu(E_i)$.
Let $K_i = co\,(\{0\} \cup C_i)$ for $i = 1, \dots, p$. Each K_i is weakly compact and convex. If $x \in \sum_{i=1}^{p} co\,(\{0\} \cup C_i)\,\mu(E_i)$ then there exist $x_i \in K_i,\ i = 1, \dots, p$, such that $x = \sum_{i=1}^{p} x_i \mu(E_i)$. Since $x_i \in K_i$ there exist $p_i \in [0,1]$ and $y_i \in C_i$ such that $x_i = p_i y_i$. Since μ is Liapounov there exists a measurable set $A_i \subseteq E_i$ such that $\mu(A_i) = p_i \mu(E_i)$. Let now $A = \bigcup_{i=1}^{p} A_i$.

$$\begin{aligned} x &= \sum_{i=1}^{p} x_i \mu(E_i) = \sum_{i=1}^{p} y_i p_i \mu(E_i) = \sum_{i=1}^{p} y_i \mu(A \cap E_i) \in \\ &\in M_\Gamma(A) \subset R_X(M_\Gamma). \end{aligned}$$

We prove now the converse inclusion. If $x \in R_X(M_\Gamma)$ then there exist a set $E \in \Sigma$ and $x_i \in C_i$ such that $x \in M_\Gamma(E)$ and then $x = \sum_{i=1}^{p} x_i \mu(E \cap E_i)$. We set

$$\alpha_i = \begin{cases} \dfrac{\mu(E \cap E_i)}{\mu(E_i)} & \text{if } \ \mu(E_i) > 0; \\ 0 & \text{if } \ \mu(E_i) = 0; \end{cases} \qquad\qquad i = 1, \dots, p.$$

Since $\alpha_i \in [0,1]$ and $x_i \in C_i$, we have that $\alpha_i x_i \in K_i$ and

$$x = \sum_{i=1}^{p} x_i \alpha_i \mu(E_i) \in \sum_{i=1}^{p} K_i \mu(E_i).$$

Therefore the range of M_Γ is the direct sum of a finite family of convex weakly compact sets and then it is convex and weakly compact. $\square$

We want to obtain now the same result for the multimeasure M_G. First of all we need an analogous result for single valued measures. What we prove in the following two results is that the indefinite integral of a vector valued simple function with respect to a non atomic measure is Liapounov and that the sum of suitable vector valued measures is Liapounov too.

Proposition 3.7 *Every simple measure m, that is every indefinite integral of a simple function, is a Liapounov measure.*

Proof: Let $f = \sum_{i=1}^{p} x_i 1_{E_i}$ and $m = m_f$. It is enough to prove that $R(m)$ is convex and closed. If $r, s \in R(m)$ then there exist $A, B \in \Sigma$ such that $r = \sum_{i=1}^{p} x_i \mu(A \cap E_i)$ and $s = \sum_{i=1}^{p} x_i \mu(B \cap E_i)$. For the sake of simplicity we denote by $A_i = A \cap E_i$, and $B_i = B \cap E_i$ for $i = 1, \dots, p$. If $t \in]0,1[$, as in Candeloro and Martellotti [3, Lemma 2.2 and Theorem 2.4], for every $A \in \Sigma$, let $(A_t)_t$ be such that $\mu(A_t) = t\mu(A),\ t \in [0,1]$, and let $C_t^i = (A_i \setminus B_i)_t \cup (A_i \cap B_i) \cup (B_i \setminus A_i)_{1-t}$. By construction we have: $C_t^i \subset E_i$ and $\mu(C_t^i) = t\mu(A_i) + (1-t)\mu(B_i)$, for every $i = 1, \cdots, p$. Let $C_t = \bigcup_{i \leq p} C_t^i$. We have

$$\begin{aligned} m(C_t) &= m(\bigcup_{i \leq p} C_t^i) = \int_{\cup_{i \leq p} C_t^i} x_i 1_{E_i} d\mu = \sum_{i=1}^{p} x_i \mu(E_i \cap C_t^i) = \\ &= \sum_{i=1}^{p} x_i \mu(C_t^i) = \sum_{i=1}^{p} x_i [t\mu(A_i) + (1-t)\mu(B_i)] = \\ &= tr + (1-t)s. \end{aligned}$$

We are now ready to prove the closedness of the range. Let $(y_k)_k$ be a sequence in $R(m)$ converging to some y_0. Since $y_k \in R(m)$ there exists $A_k \in \Sigma$ such that $y_k = \sum_{i=1}^{p} x_i \mu(A_k \cap E_i)$, for every $k \in I\!N$. We denote by A_k^i the set $A_k \cap E_i$ and by σ_k^i the number $\mu(A_k^i)$, for $i = 1, \dots, p$ and $k \in I\!N$. Since μ is a non atomic scalar measure, by Liapounov Theorem, for each $i = 1, \dots, p$, $\mu(\Sigma \cap E_i) = [0, \mu(E_i)]$. Hence, with a diagonal process, we can find a subsequence $\sigma_{k_n}^i$, and p sets $F_i \in \Sigma \cap E_i,\ i = 1, \dots, p$ such that

$$\lim_{k_n \to \infty} \sigma_{k_n}^i = \mu(F_i) \qquad i = 1, \dots, p.$$

Hence, setting $F = \bigcup_{i \le p} F_i$,

$$\lim_{k\to\infty} y_{n_k} = \sum_{i=1}^{p} x_i \mu(A_n^i) = m(F) \in R(m).\square$$

Remark 3.8 If m_1, m_2 are simple measures then the measure (m_1, m_2) is Liapounov. The proof is similar to previous one.

Theorem 3.9 *Let X, Y be two Banach spaces with X satisfying the* RNP, *μ a non atomic countably additive bounded measure, $f = \sum_{i=1}^{p} x_i 1_{E_i}$ an Y-valued, simple function, and $n_2 = \int_\cdot e d\mu$ a X-valued Liapounov measure. Then setting $n_1 = \int_\cdot f d\mu$, the range of the pair (n_1, n_2) is convex and compact in $Y \times X_w$.*

Proof: This will be done by readapting some of the arguments of Lindenstrauss's proof of Liapounov Theorem given in Lindenstrauss [11].
Let $\nu = |n_1| + |n_2|$. By Dunford Schwartz [7] Theorem III.2.20, $|n_1| = \int \|f\| d\mu$, $|n_2| = \int \|e\| d\mu$. Observe that ν is equivalent to μ.
Let $W = \{g : 0 \le g \le 1\} \subset L^\infty_\nu$, and let $T : W \to Y \times X$ be the map defined by:

$$T(g) = (T_1(g), T_2(g)) = \left(\int_\Omega g dn_1, \int_\Omega g dn_2\right).$$

W is a w^*-compact and convex subset of L^∞_ν.
Define now $\varphi(\omega) := \sum_{i=1}^{p} c_i 1_{E_i}(\omega)$ where

$$c_i = \begin{cases} \dfrac{x_i}{\|x_i\|} & \text{if } x_i \neq 0, \\ 0 & \text{otherwise.} \end{cases}$$

Then $\int_E \varphi d|n_1| = \sum_{i=1}^{p} c_i |n_1|(E \cap E_i)$ and, since $|n_1|(H) = \sum_{i=1}^{p} \|x_i\| \mu(H \cap E_i)$,

$$\int_E \varphi d|n_1| = \sum_{i=1}^{p} \frac{x_i}{\|x_i\|} \|x_i\| \mu(H \cap E_i) = \sum_{i=1}^{p} x_i \mu(E \cap E_i) = n_1(E);$$

therefore $\varphi = \dfrac{dn_1}{d|n_1|}$.

Note that each component $T_i : (L^\infty_\nu, w^*) \to \cdot_w$ is continuous since n_1 is simple, both n_i's are absolutely continuous with respect to μ and X has the RNP.

In fact if $(g_\beta)_{\beta\in\Lambda}$ is a net in L^∞_ν which is w^*-convergent to g and we denote by $\theta_1 = \dfrac{dn_1}{d|n_1|} \cdot \dfrac{d|n_1|}{d\nu}$ and $\theta_2 = -e \cdot \dfrac{d\mu}{d\nu}$ we have that $\theta_1 \in L^1_\nu(Y)$, $\theta_2 \in L^1_\nu(X)$, $\theta_i = \dfrac{dn_i}{d\nu}$, and for every $x_1^* \in X^*, x_2^* \in Y^*$,

$$\begin{aligned} x_i^* T_i(g_\beta) &= x_i^* \int_\Omega g_\beta dn_i = x_i^* \int_\Omega g_\beta \theta_i d\nu = \int_\Omega g_\beta x_i^*(\theta_i) d\nu \to \\ & \int_\Omega g x_i^*(\theta_i) d\nu = x_i^* \int_\Omega g dn_i \qquad i = 1, 2. \end{aligned} \tag{4}$$

Then, in $Y \times X$, equipped with the product of the weak topologies, $T(W)$ is compact, and therefore, closed. Moreover $T(W)$ is convex.

We prove now that $T(W) = R(n_1, n_2)$, that is for every pair $(a_1, a_2) \in T(W)$ there exists a measurable set U such that $(n_1(U), n_2(U)) = (a_1, a_2)$.

The set $W_0 = T^{-1}(\{(a_1, a_2)\})$ is convex and w^*-compact and hence it has extreme points. So it is enough to prove that if $g \in ext(W_0)$ then $g = 1_U$ for some measurable set U. Let $g \in ext(W_0)$. Assume by contradiction that there exist $\varepsilon > 0$ and $Z \in \Sigma$ such that $\mu(Z) > 0$ and $\varepsilon \le g \le 1 - \varepsilon$ on Z. Let $Z_i = E_i \cap Z$ and I be the the set $I = \{i \le p : \mu(Z_i) > 0\}$.

Let $i \in I$ be fixed. Since μ is non atomic there exists $A_i \subset Z_i$ such that $\mu(A_i) > 0$ and $\mu(Z_i \setminus A_i) > 0$. By assumption on n_2, there exist $B_i \subset A_i$, $D_i \subset Z_i \setminus A_i$ such that

$$n_2(B_i) = \frac{1}{2} n_2(A_i), \quad n_2(D_i) = \frac{1}{2} n_2(Z_i \setminus A_i).$$

Let $s_i, t_i \in I\!R$, be such that $s_i^2 + t_i^2 > 0$, $|s_i| \le \varepsilon, |t_i| \le \varepsilon$ and $s_i[\mu(A_i) - 2\mu(B_i)] = t_i[\mu(Z_i \setminus A_i) - 2\mu(D_i)]$. Let

$$h_i = \begin{cases} s_i[1_{A_i} - 2 \cdot 1_{B_i}] - t_i[1_{Z_i \setminus A_i} - 2 \cdot 1_{D_i}] & i \in I \\ 0 & \text{otherwise,} \end{cases}$$

and $h = \sum_{i=1}^p h_i 1_{E_i}$. Then easily $\int_\Omega h dn_j = 0$, $j = 1, 2$ and hence $g \pm h \in ext(W_0)$, which is a contradiction.

This shows that $R(n_1, n_2)$ is convex and compact in $Y_w \times X_w$. We shall now prove that it is indeed compact in $Y \times X_w$. Let $(A_\beta)_\beta$ be a net in Σ; then, by the $Y_w \times X_w$-compactness of $R(n_1, n_2)$, without loss of generality, we can assume that $(n_1(A_\beta), n_2(A_\beta))$ $Y_w \times X_w$-converges to $(n_1(B), n_2(B))$ for some measurable set B. From the strong compactness of $R(n_1)$ in Y, for some subnet we should have $n_1(A_{\beta_i})$ strongly converges to $n_1(B)$, and therefore the subnet $(n_1(A_{\beta_i}), n_2(A_{\beta_i}))$ converges to $(n_1(B), n_2(B))$ in $Y \times X_w$. □

A useful consequence of the previous results is the following:

Corollary 3.10 *M_Γ and M_G are Liapounov measures in $(cwk(X), h)$.*

Proof: Γ, M_Γ, G and M_G take values in the hyperspace $(cwk(X), h)$ which can be embedded, thanks to the Radström Embedding Theorem, in a suitable Banach space $(Y, \|\cdot\|)$ in such a way that the embedding is isometric. Using this fact the multifunctions Γ and G can be viewed as single valued functions in $(Y, \|\cdot\|)$.
In Proposition 3.4 it was proved that M_Γ is a multimeasure. For what concerns M_G, by Propositions 3.3 and 3.4, if $(A_n)_n$ is a sequence of pairwise disjoint Σ-measurable sets and $A = \bigcup_n A_n$ then

$$\begin{aligned} M_G(A) &= M_\Gamma(A) - \int_A e d\mu = \sum_{n=1}^{\infty} \left[M_\Gamma(A_n) - \int_{A_n} e d\mu \right] = \\ &= \sum_{n=1}^{\infty} M_G(A_n). \end{aligned}$$

Then $M_\Gamma, M_G : \Sigma \to Y$ satisfy Proposition 3.7 and Theorem 3.9 respectively. □

We are interested in the convexity and the closure of $R_X(M_G)$ in X, and not only that of $R(M_G)$.

Remark 3.11 Since Σ is a σ-algebra and X is a reflexive Banach space every vector measure $m : \Sigma \to X$ is closed in the sense of Kluvanek and

Knowles [9] (Theorem IV.7.1 of [9]). (For the definition of closedness see subsection IV.2 of [9].)

Lemma 3.12 (Lemma 7 of [10]) *Let M be a perfect multimeasure. Suppose that $S(M)$ contains a family $\mathcal{H}$ consisting of convex measures such that $\mathcal{H}$ fills out M and for any $m_1, m_2 \in \mathcal{H}$, the measure (m_1, m_2) is convex. Then $R_X(M)$ is convex.*

Theorem 3.13 (Theorem V.1.1 of [9]) *If $m : \Sigma \to X$ is a closed vector measure the following properties are equivalent:*

(3.13.1) *for every $E \notin \mathcal{N}(m)$, there exists a bounded, measurable scalar function s not vanishing on E with respect to m such that $\int_E sdm = 0$;*

(3.13.2) *m is a Liapounov measure.*

Using Lemma 3.12 and Theorem 3.13 we are able to prove that:

Theorem 3.14 *If for every $\omega \in \Omega$, $0 \notin G(\omega)$ then $R_X(M_G)$ is convex.*

Proof: By Proposition 3.4 and since $G = \Gamma - e$, M_G is a perfect multimeasure and the family $\widetilde{\mathcal{H}} = \{m+\lambda, \quad m \in \mathcal{H}\}$, where $\mathcal{H}$ is given in (3), fills out M_G. By Theorem 3.9, (m, λ) is Liapounov for every $m \in \mathcal{H}$ and by the continuity of the sum the same is true for $(m + \lambda)$.
Using Lemma 3.12 it is enough to prove the convexity of $(m_1 + \lambda, m_2 + \lambda)$ for every pair of measures in $\widetilde{\mathcal{H}}$. By Remark 3.11 $(m_1 + \lambda, m_2 + \lambda)$ is closed and, by Theorem 3.13, it is enough to prove the statement (3.13.1) for every pair $(m_1 + \lambda, m_2 + \lambda)$.
If $E \notin \mathcal{N}(m_1 + \lambda, m_2 + \lambda)$, then $E \notin \mathcal{N}(m_1 + \lambda)$ or $E \notin \mathcal{N}(m_2 + \lambda)$.
So there are just three alternatives.
If $E \notin \mathcal{N}(m_1 + \lambda)$ and $E \in \mathcal{N}(m_2 + \lambda)$ there exists a bounded, measurable scalar function s_1 which is not $(m_1 + \lambda)$-null and such that $\int_E s_1 d(m_1 + \lambda) = 0$. As $|m_2 + \lambda|(E) = 0$ clearly

$$\int_E s_1 d(m_1 + \lambda, m_2 + \lambda) = \left(\int_E s_1 d(m_1 + \lambda), \int_E s_1 d(m_2 + \lambda)\right) = (0, 0).$$

Obviously s_1 is not $(m_1+\lambda, m_2+\lambda)$-null on E. Analogously one treats the case $E \in \mathcal{N}\ (m_1+\lambda)$ and $E \notin \mathcal{N}\ (m_2+\lambda)$.
We have to check now the case $E \notin \mathcal{N}\ (m_1+\lambda)$ and $E \notin \mathcal{N}\ (m_2+\lambda)$. We remember that

$$\begin{aligned}(m_1+\lambda)(E) &= \sum_{k=1}^{p}[x_k\mu(E\cap E_k)+\lambda(E\cap E_k)],\\ (m_2+\lambda)(E) &= \sum_{k=1}^{p}[y_k\mu(E\cap E_k)+\lambda(E\cap E_k)].\end{aligned}$$

If $E \notin \mathcal{N}\ (m_1+\lambda)$ since $m_1+\lambda \ll \mu$, $E \notin \mathcal{N}\ (\mu)$. Therefore there should exist $k \in \{1,\dots,p\}$ such that $\mu(E\cap E_k) \neq 0$. From (3.13.1) there exists a bounded measurable scalar function s which is not μ-null but $\int_{E\cap E_k} sd\mu = 0$. Hence

$$\int_{E\cap E_k} sdm_1 = x_k\int_{E\cap E_k} sd\mu = 0; \qquad \int_{E\cap E_k} sdm_2 = y_k\int_{E\cap E_k} sd\mu = 0.$$

Moreover

$$\int_{E\cap E_k} sd\lambda = \int_{E\cap E_k} sed\mu = 0;$$

in fact, by Lebesgue's convergence theorem,

$$\left|\int_{E\cap E_k} sed\mu\right| = \lim_{n\to\infty}\left|\int_{E\cap E_k} se_n d\mu\right| \leq \lim_{n\to\infty}\|e_n\|_\infty\|s\|_1 = 0$$

where $e_n = e\cdot 1_{\{\omega:\|e(\omega)\|_X\leq n\}}$. Then we have that:

$$\int_{E\cap E_k} sd(m_1+\lambda, m_2+\lambda) = (0,0).$$

Let A be the support set of s in $E\cap E_k$. If s were $(m_1+\lambda)$-null then we should have

$$|m_1+\lambda|(A) = \int_A \|x_k - e(\omega)\|d\mu = 0,$$

whence $\|x_k - e(\omega)\| = 0$ μ-almost everywhere, that is $e(\omega) = x_k \in C_k = \Gamma(\omega)$, μ-a.e. in A. This means that $0 \in G(\omega) = \Gamma(\omega) - e(\omega)$ μ-a.e. in A, contradiction. So s is not $(m_1+\lambda)$-null in E and then it cannot be $(m_1+\lambda, m_2+\lambda)$-null in the same set. Then, applying Lemma 3.12, the convexity follows. $\square$

Since X is reflexive and separable, the weak topology of X induced on any ball αX_1 is metrizable, by means of the metric

$$\rho(x,y) = \sum_{n=1}^{\infty} \frac{1}{2^n} \frac{|x_n^*(x-y)|}{1+|x_n^*(x-y)|}$$

where $\{x_n^*, n \in I\!N\}$ is a fixed dense subset of X_1^*.
Therefore the Hausdorff topology on $cwk(\alpha X_1)$ defined by means of the weak topology of αX_1, coincides with the Hausdorff metric topology h_ρ induced by ρ (Christensen [5] pag. 52).
Since $\rho(x-y) \leq \|x-y\|$, for every pair of bounded sets $A, B \subset X$

$$h_\rho(A,B) \leq h(A,B) \tag{5}$$

We remind a result due to Christensen, in the formulation that we shall need. Afterwords

Theorem 3.15 (3.1 of [5]) *Let $k(\alpha X_1)$ be the hyperspace of compact subset of αX_1, equipped with the Hausdorff topology. A closed set $\mathcal{R}$ in $k(\alpha X_1)$ is compact if and only if the set $\cup_{K\in\mathcal{R}} K$ is compact in the space αX_1.*

Theorem 3.16 *$R_X(M_G)$ is weakly compact.*

Proof: Thanks to Theorem 3.15 it is enough to prove that $R(M_G)$ is compact in $(cwk(\alpha X_1), h_\rho)$ where $\alpha = \mu(\Omega) \cdot \max\limits_{i=1,\dots,n} h(C_i, \{0\}) + r$ and r is a positive number such that $R(\lambda) \subset rX_1$.
In order to prove this we consider the pair (M_Γ, λ). We have already proved that the first is a simple valued measure in $Y = (cwk(X), h)$ and the second a Liapounov measure in X. Applying Theorem 3.9 to (M_Γ, λ) we obtain that the range of the pair is compact in $(cwk(X), h) \times X_w$.
We consider now the map $\varphi : (cwk(X), h) \times (\alpha X_1, \rho) \to (cwk(X), h) \times (cwk(\alpha X_1), h_\rho)$ defined by $\varphi(C, x) = (C, \{x\})$. Since the map $x \mapsto \{x\}$ is an isometry of $(\alpha X_1, \rho)$ into $(cwk(\alpha X_1), h_\rho)$ we obtain the continuity of φ, and therefore $\varphi(R(M_\Gamma, \lambda))$ is compact in $(cwk(X), h) \times (cwk(\alpha X_1), h_\rho)$.
Moreover we can observe that the set $R(M_\Gamma)$ is compact in $(cwk(X), h)$, since it is the convex hull of a finite set; hence, by (5), $R(M_\Gamma)$ is compact in

$(cwk(X), h_\rho)$. Also since $M_\Gamma \subset \alpha X_1$ for every $E \in \Sigma$, we conclude that $R(M_\Gamma)$ is compact in $(cwk(\alpha X_1), h_\rho)$.
Finally, from (5), $\varphi(R(M_\Gamma, \lambda))$ is compact in $(cwk(\alpha X_1), h_\rho)^2$.
Since the sum in $(cwk(\alpha X_1), h_\rho)$ is h_ρ-continuous, this shows that $R(M_G)$ is h_ρ-compact and concludes the proof. $\square$

A useful consequence of the previous theorems is:

Theorem 3.17 *Let F be a measurable multifunction defined by $F = G \cup \{0\} = (\Gamma - e) \cup \{0\}$ where Γ takes values in $cwk(X)$. If $0 \notin G$ then, for every $E \in \Sigma$, $(A) - \int_E F d\mu$ is convex and weakly compact.*

Proof: It is enough to prove that

$$(A) - \int_E F d\mu = R_X(M_G|_{E \cap \Sigma}).$$

We shall prove the last equality only in the case $E = \Omega$. Let $z \in (A) - \int_\Omega F d\mu$: then there exists $f \in S^1_F$ such that $\int_\Omega f d\mu = z$. Let H be the support of f. The function $f \cdot 1_H \in S^1_{G \cdot 1_H}$ and

$$z = \int_\Omega f d\mu = \int_H f d\mu \in (A) - \int_H G d\mu = M_G(H).$$

Conversely, if $z \in M_G(K)$ for some measurable set K; then $z \in (A) - \int_K G d\mu$. If $s \in S^1_{G \cdot 1_K}$ is such that $z = \int_K s d\mu$, then $z = \int_\Omega s 1_K d\mu \in (A) - \int_\Omega F d\mu$. $\square$

3.2. Integrands with unbounded values

We now turn to the general case, namely, assume that $C_i \in cf(X)$, $i = 1, \dots, p$ and $0 \notin G$. As before, consider $F = G \cup \{0\} = (\Gamma - e) \cup 0$.

Proposition 3.18 *For every $E \in \Sigma$, $(A) - \int_E F d\mu$ is convex and it is the union an increasing sequence of weakly compact sets.*

Proof: We denote by Γ_n and F_n the multifunctions:

$$\Gamma_n(\omega) = \Gamma(\omega) \cap n X_1, \qquad F_n(\omega) = (\Gamma_n(\omega) - e(\omega)) \cup \{0\}.$$

As Γ is simple, there should exists $\overline{n} \in I\!N$ such that for every $\omega \in \Omega$, $\Gamma_n(\omega) \neq \emptyset$, for every $n \geq \overline{n}$. We shall consider only $n \geq \overline{n}$.
Moreover, since Γ_n takes values in $cwk(X)$ for every $n \in I\!N$, by Theorem 3.17, $(A) - \displaystyle\int_E F_n d\mu$ is convex and weakly compact for every $n \geq \overline{n}$.
The assertion will follow from the equality

$$(A) - \int_E F d\mu = \bigcup_{n \geq \overline{n}} (A) - \int_E F_n d\mu. \tag{6}$$

and the obvious inclusion

$$(A) - \int_E F_n d\mu \subset (A) - \int_E F_{n+1} d\mu.$$

We will prove the result just for $E = \Omega$. Obviously

$$\bigcup_{n \geq \overline{n}} (A) - \int_\Omega F_n d\mu \subset (A) - \int_\Omega F d\mu$$

since $S^1_{F_n} \subset S^1_F$ for every $n \geq \overline{n}$. Viceversa let $x \in (A) - \displaystyle\int_\Omega F d\mu$; then there exists $f \in S^1_F$ such that $x = \displaystyle\int_\Omega f d\mu$. We denote by S the support of f. Then $x = \displaystyle\int_S f d\mu$ and, for every $\omega \in S$ it is $f(\omega) \in \Gamma(\omega) - e(\omega)$. Then clearly $\varphi = f + e \in S^1_\Gamma$ and

$$x = \int_S \varphi d\mu - \int_S e d\mu.$$

In general φ is not simple, but we shall construct a simple function $g \in S^1_\Gamma$ such that $\int_S g d\mu = \int_S \varphi d\mu$.
Without loss of generality we can suppose that for every k the set $S \cap E_k$ is of positive μ-measure, otherwise let $I = \{i_1, \ldots, i_k\}$ be the set of indexes such that $\mu(S \cap E_i) = 0$ for $i \in I$,

$$\widetilde{S} = S \setminus \bigcup_{k \in I} (E_k : \mu(S \cap E_k) = 0)\,;$$

then we can replace S with $\widetilde{S}$. It is:

$$\int_S \varphi d\mu \;=\; \sum_{k=1}^{p} \int_{S \cap E_k} \varphi d\mu = \sum_{k=1}^{p} \frac{\displaystyle\int_{S \cap E_k} \varphi d\mu}{\mu(S \cap E_k)} \mu(S \cap E_k).$$

Define

$$x_k = \frac{\displaystyle\int_{S\cap E_k} f d\mu}{\mu(S\cap E_k)} \in C_k,$$

and set $g(\omega) = x_k$ for every $\omega \in S\cap E_k$, for $k = 1, 2, \ldots, p$ and $g(\omega) = 0$ otherwise.

Let $n_x = \max\{[\|x_1\|], \cdots, [\|x_n\|], \overline{n}\}$. The simple function g is a selection of Γ_{n_x} and has the same integral of φ. Then $(g-e)1_S$ is an integrable selection of F_{n_x}. This proves that

$$(A) - \int_\Omega F d\mu \subset \bigcup_{n\geq\overline{n}} (A) - \int_\Omega F_n d\mu. \Box$$

Remark 3.19 Note also that the equality (6) in the proof has been derived without making use of the hypothesis $0 \notin G$; the last assumption indeed has been used only to apply Theorem 3.17 to each F_n.

3.3. Integrands which may contain the origin

We now turn to the general case, namely we consider possibly unbounded integrands which may contain the origin.

Theorem 3.20 *Let $F : \Omega \to cf(X)$ be a measurable multifunction of the following type: $F = (\Gamma - e) \cup \{0\}$, where Γ is simple and takes values in $cf(X)$ and $e \in L^1_\mu(X)$ has Liapounov indefinite integral. Then, for every $E \in \Sigma$, $(A) - \int_E F d\mu$ is convex and it is a countable union of an increasing sequence of weakly compact sets.*

Proof: We denote by Ω_0 the set $\{\omega \in \Omega :\ 0 \in G(\omega) = \Gamma(\omega) - e(\omega)\}$.
The map $\omega \mapsto d(0, G(\omega))$ is measurable (since G is Effros measurable); then $\Omega_0 = \{d(G, 0) = 0\} \in \Sigma$.
Let F_n and Γ_n be as in the proof of Theorem 3.18. Note that, since $0 \in F(\omega)$, S^1_F is a decomposable subset of $L^1_\mu(X)$. Therefore, for every $E \in \Sigma$, using Remark 3.19, we have:

$$\begin{aligned}(A) - \int_E F d\mu &= (A) - \int_{E\cap\Omega_0} F d\mu + (A) - \int_{E\setminus\Omega_0} F d\mu = \\ &= \bigcup_{n\geq\overline{n}} (A) - \int_{E\cap\Omega_0} F_n d\mu + (A) - \int_{E\setminus\Omega_0} F d\mu.\end{aligned}$$

Now in every measurable subset of Ω_0 we have that $F_n(\omega) = \Gamma_n(\omega) - e(\omega) \in cwk(X)$. Then, by the main theorem of Byrne [2], $(A)-\int_{E\cap\Omega_0} F_n d\mu \in cwk(X)$ for every $n \geq \overline{n}$. Again $\left((A) - \int_{E\cap\Omega_0} F_n d\mu\right)_n$ is an increasing sequence, and so its union is convex, while $(A) - \int_{E\setminus\Omega_0} F d\mu$ is convex and it is a countable union of an increasing sequence of weakly compact sets by Theorem 3.14.
In conclusion $(A) - \int_E F d\mu$ is convex, furthermore it is clearly the union

$$(A) - \int_E F d\mu \quad = \bigcup_{n \geq \overline{n}} (A) - \int_E F_n d\mu$$

of an increasing sequence of weakly compact sets. □

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