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On the COMPARISON of AUMANN and BOCHNER INTEGRALS[1]

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Abstract

We introduce the Aumann integral in the finitely additive setting and we compare it with the Bochner integral.

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1 Introduction

In Mathematical Economics the search for Walras equilibria makes use of the multivalued integral as defined in 1962 by Aumann, [2]. In the classical model the bundle set, the commodity space and so on are defined as subsets or functions in the Euclidean space $\mathbb{R}^n$. However, *in a world of uncertainty where there are an infinite number of states or an intertemporal economy having an infinite number of time periods* (*e.g. an infinite horizon*), *the appropriate model for the space of commodities is an infinite dimensional vector space,* [1].
There exists a large literature concerning the integral of multifunctions with values in infinite dimensional vector spaces, and its applications in Mathematical Economics: nevertheless the finitely additive setting has been quite neglected so far. This is in a certain sense rather surprising, for the finitely additive case could be the best solution to a basic conflict that the countable additive setting leaves unsolved: indeed, it is known that in the model of a large economy it is desirable to assign zero influence to a single agent, and therefore to equip the space of the agents Ω with the structure of a non atomic measure space (Ω, Σ, μ).
Moreover, *no group of agents is a priori excluded from forming a coalition. Indeed, for a simple economy we defined the set of coalitions to be the set of all subsets. If the economy is not simple the σ-algebra Σ is introduced only for technical reasons. Conceptually Σ should be considered as*

[1]Lavoro svolto nell' ambito del G.N.A.F.A. del C.N.R.

the set of all subset, [13]. The *technical reasons* lie in the famed Ulam Theorem, [5], *Assuming the C.H., if* $card(\Omega) = c$ *every non atomic measure on* $\mathcal{P}\ (\Omega)$ *is identically zero.*
This is not the case for strongly non atomic finitely additive measures.
In this paper we shall examine some aspects of multivalued integration in a separable Banach space X with respect to a scalar finitely additive measure.
Obviously the assumption $\Sigma = \mathcal{P}(\Omega)$ would substantially semplify measurability questions, but, again because of the Ulam Theorem, would not generalize the countably additive case: for this reason our investigation has been accomplished with any σ-algebra Σ on Ω.
The point of view of this paper is that of comparing the Aumann approach with the more classical Bochner approach, namely using approximation via simple multifunctions. In fact, if Aumann integral is the more natural in view of the applications in Mathematical Economics, in general it lacks most of the properties one would expect an integral to enjoy, as for example convergence results.
Hence the equivalence of the Aumann integral with a classical Bochner integral would greatly enhance its properties. This equality is known in the countably additive case for measurable integrands with convex, compact values in [8, 12] and for totally measurable integrands with convex, weakly compact values in [6]. Here we will obtain the equivalence in the finitely additive setting for totally measurable integrands with convex, compact values: the main idea is that of using Stone extensions which preserve the Bochner integral. Therefore the comparison of the Aumann and the Bochner integral has been transferred to the comparison of the Aumann integral and the Aumann integral of the Stone extension.

2 Notations and Preliminaries

We will use the terminology of [9] and that of [7]. In particular we will use the following definitions and notations.

- X is a separable Banach space.
- X^* is the topological dual of X, X^*_s (resp. X^*_b) is the vector space X^* equipped with the $\sigma(X^*, X)$ (resp. norm) topology.
- X_1 (resp. X^*_1) is the closed unit ball in X (resp. X^*_b).
- $cb(X)$ (resp. $ck(X)$) is the collection of all non empty convex closed bounded (resp. convex compact) subsets of X.
- If A and B are subsets of X, the *excess* of A over B is

$$e(A, B) = \sup\{d(a, B) : a \in A\}$$

and the *Hausdorff distance* between A and B is

$$h(A, B) = \max\{e(A, B), e(B, A)\}.$$

Observe that for every pair of bounded sets A, B, and for every pair of elements $x, y \in X$ it is:

$$d(x, A) \leq \|x - y\| + d(y, B) + h(A, B). \tag{1}$$

- The excess $e(A, \{0\})$ is denoted by $|A|$. Then

$$|A| = \sup\{\|a\| : a \in A\}.$$

- (Ω, Σ, m) is a measurable space with Σ complete (i.e. Σ contains all m-null sets) and $m : \Sigma \to I\!R_0^+$ is a bounded finitely additive measure.
- A function $f : \Omega \to X$ will be said Σ-*measurable* provided $f^{-1}(A) \in \Sigma$ for every open set A of X.
- A function $f : \Omega \to X$ is *totally measurable* when there exists a sequence of measurable simple functions $(f_n)_n$ which converges to f in m-measure.
- $TM(\Omega, \Sigma, m, X)$ is the space of all totally measurable functions $f : \Omega \to X$.

Remark 2.1 Observe that any $f \in TM(\Omega, \Sigma, m, X)$ is also Σ-measurable. This is a consequence of the σ-completeness of Σ and of the null completeness of m. An elementary proof of this fact can be given using Theorem 2.4 of [3] in order to prove that a totally measurable function f is measurable in the Greco sense with respect to the m^*-closure of Σ and using Proposition 1.7.b of [4] to observe that, if Σ is σ-complete and m is null complete the m^*-closure of Σ is exactly Σ. Finally since Σ is a σ-algebra, the Greco measurability coincides with the usual one. This result holds for scalar functions, but it applies also to $f \in TM(\Omega, \Sigma, m, X)$.

- A multifunction $F : \Omega \to ck(X)$ is *Effros-measurable* (shortly-measurable) if the set $F^-U = \{\omega \in \Omega : F(\omega) \cap U \neq \emptyset\}$ belongs to Σ for any open subset U of X.
- A measurable multifunction $F : \Omega \to ck(X)$ is *totally measurable* if there exists a sequence of simple measurable multifunctions $(F_n)_n$ with values in $ck(X)$ such that, for every $\alpha > 0$,

$$\lim_n m(\omega \in \Omega : h(F_n(\omega), F(\omega)) > \alpha) = 0.$$

We denote by $\mathcal{TM}(\Omega, \Sigma, m, ck(X))$ the space of totally measurable multifunctions with non empty, compact convex values.
- A totally measurable multifunction $F : \Omega \to ck(X)$ is (B)-*integrable* if there exists a sequence $(F_n)_n$ of simple multifunctions, F_n with values in $ck(X)$, such that $h(F_n, F)$ m-converges to zero and moreover $\lim_{k,n\to\infty} \int_\Omega h(F_k, F_n) dm = 0$. In this case we shall say that $(F_n)_n$ is a *defining sequence* for F and we define the (B)-*integral* of F over E as

$$(B) - \int_E F dm := \lim_{n\to\infty} (B) - \int_E F_n dm.$$

We denote by $L^1(\Omega, \Sigma, m, ck(X))$ the space of all (B)-integrable multifunctions.

3 The Stone transform

Let (Ω, Σ) be a measurable space and $m : \Sigma \to I\!R_0^+$ be a bounded finitely additive measure. Let S be the Stone space associated to Σ, $\mathcal{G}$ the algebra of clopen sets of S and $\tau : \Sigma \to \mathcal{G}$ the Stone isomorphism. $\mathcal{G}_\sigma$ is the σ-algebra on S generated by $\mathcal{G}$.
We denote by $\overline{m} : \mathcal{G}_\sigma \to I\!R_0^+$ the Stone extension of m, [19] and by $(S, \mathcal{G}_\sigma, \overline{m})$ the Stone space relative to (Ω, Σ, m).
The natural injection from Σ onto $\mathcal{G}_\sigma$ induces an isometric isomorphism from $\mathrm{cl}\{TM(\Omega, \Sigma, m, X)\}$ onto $TM(S, \mathcal{G}_\sigma, \overline{m}, X)$ and from $\mathrm{cl}\{L^1(\Omega, \Sigma, m, X)\}$ onto $L^1(S, \mathcal{G}_\sigma, \overline{m}, X)$ (see for instance [10]). This isomorphism preserves the order and other structures on $L^1(\Omega, \Sigma, m, X)$. We remind that

a) $\|\overline{f}\| = \overline{\|f\|}$ $\overline{m}$-a.e.;

b) $\overline{m}(\{s : |\overline{f}(s)| > \alpha\}) \leq m(\{\omega : |f(\omega)| \geq \alpha\}) \leq \overline{m}(\{s : |\overline{f}(s)| \geq \alpha\})$;

c) if $f \in L^1(m)$, then $\overline{f} \in L^1(\overline{m})$ and for every $E \in \Sigma$

$$\int_E f dm = \int_{\tau(E)} \overline{f} d\overline{m}.$$

It is well known that $(ck(X), h)$ is a separable complete metric space. Moreover, using Theorem 2 of [17], $(ck(X), h)$ can be embedded as a closed convex cone in a separable Banach space $(Y, \|\cdot\|_Y)$ in such a way that the embedding is isometric and the addition and the multiplication by a non negative real number in Y induce the corresponding operations in $ck(X)$.
Now using this fact, and the construction of the single valued vector case, we can consider the Stone transform of a totally measurable multifunction $F : \Omega \to ck(X)$.
Observe that $h(\overline{F}, \{0\}) = \overline{h(F, \{0\})}$. In fact, since the embedding is isometric,

$$h(\overline{F}, \{0\}) = \|\overline{F}\|_Y = \overline{\|F\|_Y} = \overline{h(F, \{0\})}.$$

Moreover $\overline{F}$ has compact convex values $\overline{m}$-a.e..
Now, for every $F \in L^1(\Omega, \Sigma, m, ck(X))$ we can consider its extended multifunction $\overline{F}$. If $(F_n)_n = (\sum_{i=1}^{k_n} C_i^n 1_{E_i^n})$ is a defining sequence for F, then for every $E \in \Sigma$ we have:

$$\begin{aligned} (B) - \int_E F dm &= \lim_n (B) - \int_E F_n dm = \lim_n \sum_{i=1}^{k_n} C_i^n m(E \cap E_i^n) = \\ &= \lim_n \sum_{i=1}^{k_n} C_i^n \overline{m}(\tau(E) \cap \tau(E_i^n)) = \lim_n (B) - \int_E \overline{F}_n d\overline{m} = (B) - \int_E \overline{F} d\overline{m}. \end{aligned} \tag{2}$$

So $\overline{F} \in L^1(S, \mathcal{G}_\sigma, \overline{m}, ck(X))$ and the Bochner integrals of F and $\overline{F}$ agree.

4 The Aumann integral

Given a multifunction $F : \Omega \to cb(X)$ let

$$\begin{aligned} S_F &= \{f \in TM(\Omega, \Sigma, m, X) : f(\omega) \in F(\omega) \quad m - \text{a.e.}\}; \\ S_F^1 &= \{f \in S_F : f \in L^1(\Omega, \Sigma, m, X)\}. \end{aligned}$$

Definition 4.1 If F is such that S_F^1 is non empty then, for every $E \in \Sigma$ we define the *Aumann integral* (shortly (A)-*integral*) as

$$(A) - \int_E F dm = \{\int_E f dm, f \in S_F^1\}.$$

So in order to define the Aumann integral for a multifunction F with respect to a finitely additive measure m we need to prove that $S_F^1 \neq \emptyset$.

The following theorem can be found in several versions in the literature (for instance [15], [14], [7], [12], [16]).

Proposition 4.2 *Let $(T, \mathcal{T})$ be a measurable space, X a separable metric space, and F map T to non empty complete subsets of X. Then the following properties are equivalent:*

a) *for each open set U, $U \subset X, F^{-1}U \in \mathcal{T}$;*

b) *for each $x \in X$ the function $\omega \mapsto d(x, F(\omega))$ is measurable;*

c) *there exists a sequence of measurable functions $f_n : \Omega \to X$ such that $F(\omega) = cl\{f_n(\omega)\}$. In this case the sequence $(f_n)_n$ is called a* Castaing representation of F.

Now analogously to [15], and Theorem III.6 of [7] we prove that for suitable measurable multifunctions F the set

$$S_F = \{f : \Omega \to X, f \in TM(\Omega, \Sigma, m, X), f(\omega) \in F(\omega)\ m - \text{a.e.}\}$$

is non empty.

Theorem 4.3 *Let $F : \Omega \to ck(X)$ be a measurable multifuction such that $cl\{F(\Omega)\} = cl\{\bigcup_{\omega \in \Omega} F(\omega)\}$ is a compact subset of X; then $S_F \neq \emptyset$.*

Proof: Since X is separable there exists $D = \{x_n\}_n$ such that $cl\{D\} = X$. We want to construct a sequence of simple functions $(f_p)_p$ such that, for every $p \in I\!N$ and for every $\omega \in \Omega$,

$$d(f_p(\omega), F(\omega)) \leq \frac{1}{2^p} \tag{3}$$

$$\|f_{p+1}(\omega) - f_p(\omega)\| \leq \frac{1}{2^{p+1}}. \tag{4}$$

Consider $\{x_n + X_1, x_n \in D\}$; by the total boundedness of $cl\{F(\Omega)\}$, there exists $\{x_1^0, \cdots, x_{n_0}^0\} \subset D$ such that $\cup_{j=1}^{n_0}(x_j^0 + X_1) \supset cl\{F(\Omega)\}$. For every $\omega \in \Omega$ let

$$j_0(\omega) = \min\{j : j \leq n_0, F(\omega) \cap (x_j^0 + X_1) \neq \emptyset\}.$$

We set $f_0(\omega) = x_{j_0(\omega)}^0$. f_0 is a measurable simple function: in fact, for every $j = 1, \cdots, n_0$

$$f_0^{-1}(x_j^0) = \{\omega \in \Omega : F(\omega) \cap (x_j^0 + X_1) \neq \emptyset\} \setminus \cup_{k=1}^{j-1}\{\omega \in \Omega : F(\omega) \cap B(x_k^0 + X_1) \neq \emptyset\}.$$

Consider now $\{(x_n + 2^{-1}X_1), x_n \in D\}$; analogously to the previous step there exists $\{x_1^1, \cdots, x_{n_1}^1\}$ $\subset D$ such that $\cup_{j=1}^{n_1}(x_j^1 + 2^{-1}X_1) \supset cl\{F(\Omega)\}$.
Let $\Omega_i = f_0^{-1}(x_i^0)$, for $i = 1, \cdots, n_0$. For every $\omega \in \Omega$ there exists $i \leq n_0$ such that $\omega \in \Omega_i$, so $F(\omega) \cap (x_i^0 + X_1) \neq \emptyset$ and we set:

$$j_1(\omega) = \min\{j : j \leq n_0, F(\omega) \cap (x_i^0 + X_1) \cap B(x_j^1 + 2^{-1}X_1) \neq \emptyset\}.$$

So, for every $\omega \in \Omega_i$ we set $f_1(\omega) = x_{j_1(\omega)}^1$. Again f_1 is a simple measurable function.
By recurrence we can costruct f_n which satisfies (3) and (4). Moreover from (4) we obtain that $(f_n)_n$ is uniformly Cauchy in X which is complete. So the limit f of f_n exists and $f(\omega) \in F(\omega)$ since $F(\omega)$ is closed. This proves that f is totally measurable and $f \in S_F$. Observe that, since f is the uniform limit of Σ-measurable functions, f is Σ-measurable. $\square$

Definition 4.4 A multifunction $F : \Omega \to cb(X)$ is *integrably bounded* if there exists a non negative $g \in L^1(\Omega, \Sigma, m, I\!R_0^+)$ such that:

$$h(F(\omega), \{0\}) \leq g(\omega) \qquad m - \text{a.e.}$$

Remark 4.5 If $F \in \mathcal{TM}(\Omega, \Sigma, m, ck(X))$ is such that $cl\{F(\Omega)\}$ is a compact subset of X then F is integrably bounded and hence every totally m-measurable selection f is also m-integrable and so S_F^1 is non-empty and uniformly integrable.

5 Comparison between Aumann and Bochner integrals

The notations we shall use here are the same as in Section 3. From now on we suppose that $F \in \mathcal{TM}(\Omega, \Sigma, m, ck(X))$, and $cl\{F(\Omega)\}$ is a compact subset of X.
Let $j : TM(\Omega, \Sigma, m, X) \to TM(S, \mathcal{G}_\sigma, \overline{m}, X)$ be the function defined by

$$j(f) = \overline{f}.$$

We observe that:

Theorem 5.1 *Let F be integrably bounded and $cl\{F(\Omega)\}$ compact and $f \in L^1(\Omega, \Sigma, m, x)$. If for every $\alpha > 0$, $m(\{\omega \in \Omega : d(f, F) \geq \alpha\}) = 0$, then $\overline{f} \in S_{\overline{F}}^1$.*

Proof: Let $f \in L^1(\Omega, \Sigma, m, X)$ be such that for every $\alpha > 0$, $m(\{\omega \in \Omega : d(f, F) \geq \alpha\}) = 0$. Let $(f_n)_n$ be a sequence of simple functions which m-converges to f and $(F_n)_n$ a sequence of simple multifunctions m-converging to F.
Let $\gamma_n : \Omega \to I\!\!R_0^+$ be the function defined as follows:

$$\gamma_n(\omega) = d(f_n(\omega), F_n(\omega)).$$

$(\gamma_n)_n$ is a sequence of simple functions which m-converges to 0; in fact, by (1),

$$d(f_n, F_n) \leq \|f_n - f\| + d(f, F) + h(F, F_n)$$

and setting, for every $\alpha > 0$,

$$\begin{array}{rcl} A_n & = & \{\omega \in \Omega : \gamma_n(\omega) > \alpha\} \\ A_n^{'} & = & \{\omega \in \Omega : \|f_n(\omega) - f(\omega)\| > \dfrac{\alpha}{2}\} \\ A_n^{"} & = & \{\omega \in \Omega : h(F_n(\omega), F(\omega)) > \dfrac{\alpha}{2}\} \end{array}$$

we have $m(A_n) \leq m(A_n^{'}) + m(A_n^{"})$ and so $m(A_n)$ converges to 0. Using now b) of section 3, $(\overline{\gamma}_n)_n$ $\overline{m}$-converges to 0. Whithout loss of generality assume that f_n and F_n have the same representation, namely

$$f_n = \sum_{i=1}^{p_n} c_i^n 1_{E_i^n}, \qquad F_n = \sum_{i=1}^{p_n} C_i^n 1_{E_i^n};$$

since F_n has compact values, for every $i = 1, \cdots, p_n$ there exists $x_i^n \in C_i^n$ such that $d(c_i^n, C_i^n) = \|x_i^n - c_i^n\|$. Let $t_n = \sum_{i=1}^{p_n} x_i^n 1_{E_i^n}$. Observe that $t_n \in S^1_{F_n}$ and $\|t_n - f_n\| = \gamma_n$. Let

$$\overline{t}_n = \sum_{i=1}^{p_n} x_i^n 1_{\tau(E_i^n)}, \qquad \overline{f}_n = \sum_{i=1}^{p_n} c_i^n 1_{\tau(E_i^n)}.$$

We have

$$t_n - f_n = \sum_{i=1}^{p_n} (x_i^n - c_i^n) 1_{E_i^n}, \qquad \overline{t}_n - \overline{f}_n = \sum_{i=1}^{p_n} (x_i^n - c_i^n) 1_{\tau(E_i^n)},$$

and so,

$$\|\overline{t}_n - \overline{f}_n\| = \|\overline{t_n - f_n}\| = \overline{\|t_n - f_n\|} = \overline{\gamma}_n.$$

Let now $(\overline{f}_{n_k})_k, (\overline{F}_{n_k})_k, (\overline{\gamma}_{n_k})_k$ be three subsequences converging respectively to $\overline{f}, \overline{F}, 0$ $\overline{m}$-a.e.; we obtain, again using (1),

$$d(\overline{f}, \overline{F}) \leq \|\overline{f} - \overline{f}_{n_k}\| + \|\overline{f}_{n_k} - \overline{t}_{n_k}\| + d(\overline{t}_{n_k}, \overline{F}_{n_k}) + h(\overline{F}_{n_k}, \overline{F}),$$

so $\overline{f} \in S^1_{\overline{F}}$. $\square$

Observe that Theorem 5.1 holds in particular for $f \in S^1_F$.

Remark 5.2 If λ is a countably additive measure and G is an integrably bounded, compact convex valued multifunction, then the Aumann integral $(A)-\int_{\cdot} G d\lambda$ is compact and convex (see for example [12]). So in particular, for $G=\overline{F}$ and $\lambda=\overline{m}$,

$$(A)-\int_{\cdot}\overline{F}d\overline{m}=\left\{\int_{\cdot}\phi d\overline{m}:\phi\in S^1_{\overline{F}}\right\}$$

is convex and compact in X.

Corollary 5.3 If $F\in\mathcal{TM}(\Omega,\Sigma,m,ck(X))$ and $S^1_F\neq\emptyset$, then

$$(A)-\int_E Fdm\subset(A)-\int_{\tau(E)}\overline{F}d\overline{m}=(B)-\int_{\tau(E)}\overline{F}d\overline{m}=(B)-\int_E Fdm. \tag{5}$$

Proof: it is a consequence of (c) given in Section 3, Remark 5.2, Theorem 4.5 of [12] and (2).

Now we want to show that every selection of $\overline{F}$ is the Stone trasform of a function f such that $d(f,F)$ is a m-null function in the sense of [9]. To obtain this result we need the completeness of the space $L^1(\Omega,\Sigma,m,X)$. A characterization of the completeness of $L^1(\Omega,\Sigma,m,X)$ is given in [11], and related results can be found in [4].

Theorem 5.4 *Suppose that $L^1(\Omega,\Sigma,m,X)$ is complete. Let $\phi\in S^1_{\overline{F}}$, then there exists a function $f\in L^1(\Omega,\Sigma,m,X)$ such that $\overline{f}=\phi$ and, for every $\alpha>0$, $m(\{\omega\in\Omega:d(f(\omega),F(\omega))\geq\alpha\})=0$.*

Proof: Let $(\phi_n)_n$ be a defining sequence for ϕ.

$$\phi_n=\sum_{j=1}^{p_n}x_j^{(n)}1_{E_j}^{(n)},\qquad E_j^{(n)}\in\mathcal{G}_\sigma$$

for every $j=1,\cdots,p_n$.
$\mathcal{G}$ is dense in $\mathcal{G}_\sigma$ with respect to the (FN)-pseudometric defined by:

$$d_{\overline{m}}(E,F)=\overline{m}(E\Delta F),$$

for every $E,F\in\mathcal{G}_\sigma$. Then, for every $\varepsilon>0$, and for every $E\in\mathcal{G}_\sigma$, there exists $A\in\mathcal{G}$ such that $\overline{m}(A\Delta E)\leq\varepsilon$.
Let $\varepsilon_n\downarrow 0$ and let $n\in I\!\!N$ be fixed; for every $j=1,\cdots,p_n$ there exists $A_j^{(n)}\in\mathcal{G}$ such that

$$\overline{m}(A_j^{(n)}\Delta E_j^{(n)})\leq\frac{\varepsilon_n}{\sum_{i=1}^{p_n}\|x_i^{(n)}\|}.$$

Let $g_n=\sum_{i=1}^{p_n}x_i^{(n)}1_{A_i}^{(n)}$; g_n is $\mathcal{G}$-simple and

$$\int_S\|g_n-\phi_n\|d\overline{m}\leq\sum_{i=1}^{p_n}\|x_i^{(n)}\|\overline{m}(A_i^{(n)}\Delta E_i^{(n)})\leq\varepsilon_n. \tag{6}$$

Let $B_i^{(n)} = \tau^{-1}(A_i^{(n)})$ and $\gamma_n = \sum_{i=1}^{p_n} x_i^{(n)} 1_{B_i}^{(n)}$. So g_n is the Stone transform of γ_n.
Since $(\phi_n)_n$ is defining we find from (6):

$$\lim_n \int_S \|g_n - \phi\| d\overline{m} = 0,$$

namely g_n converges to ϕ in $L^1(S, \mathcal{G}_\sigma, \overline{m}, X)$. So, $(\gamma_n)_n$ is Cauchy in $L^1(\Omega, \Sigma, m, X)$. Since m is self-separable then $L^1(\Omega, \Sigma, m, X)$ is complete and so there exists a function $f \in L^1(\Omega, \Sigma, m, X)$ such that γ_n converges to f in $L^1(\Omega, \Sigma, m, X)$. It follows from Remark 2.1 that f is Σ-measurable.
The sequence $(g_n)_n$ converges to ϕ in L^1 and thus $\overline{f} = \phi$ $\overline{m}$-a.e..
It only remains to prove that for every $\alpha > 0$ $m(\{\omega \in \Omega : d(f, F) \geq \alpha\}) = 0$.
Let $(F_n)_n$ be a defining sequence for F; then $(\overline{F}_n)_n$ is $\mathcal{G}$-measurable and defining for $\overline{F}$.
By the countable additivity of $\overline{m}$, we can consider two subsequences $(g_{n_k})_k, (\overline{F}_{n_k})_k$ of $(g_n)_n$ and $(\overline{F}_n)_n$ respectively which converge to $\overline{f}$ and $\overline{F}$, $\overline{m}$-a.e.
Let now $k \in I\!N$ be fixed; we can represent $(g_{n_k})_k$ and $(\overline{F}_{n_k})_k$ with the same $\mathcal{G}$-measurable decomposition of S

$$(g_{n_k})_k = \sum_{i=1}^{p_{n_k}} x_i^{(n_k)} 1_{\tau(E_i^{(n_k)})} \qquad (\overline{F}_{n_k})_k = \sum_{i=1}^{p_{n_k}} C_i^{(n_k)} 1_{\tau(E_i^{(n_k)})}.$$

Let $c_i^{(n_k)} \in C_i^{(n_k)}$ such that $\|x_i^{(n_k)} - c_i^{(n_k)}\| = d(x_i^{(n_k)}, C_i^{(n_k)})$, and $t_{n_k} = \sum_{i=1}^{p_{n_k}} c_i^{(n_k)} 1_{\tau(E_i^{(n_k)})}$. Then $\|t_{n_k} - g_{n_k}\| = d(g_{n_k}, \overline{F}_{n_k})$ whence

$$\lim_k \|t_{n_k} - \overline{f}\| = 0 \quad \overline{m} - \text{a.e.}.$$

Then, for every $\alpha > 0$

$$\lim_k \overline{m}(\|t_{n_k} - \overline{f}\| \geq \alpha) = 0.$$

Let $\sigma_{n_k} = \sum_{i=1}^{p_{n_k}} c_i^{(n_k)} 1_{E_i^{(n_k)}}$. By b) of section 3, $\lim_k m(\|\sigma_{n_k} - f\| \geq \alpha) = 0$ for every $\alpha > 0$.
Since

$$d(f, F) \leq \|f - \sigma_{n_k}\| + d(\sigma_{n_k}, F) \leq \|f - \sigma_{n_k}\| + h(F_{n_k}, F)$$

we have

$$m(d(f, F) \geq \alpha) \leq \lim_k \left[m(\|f - \sigma_{n_k}\| \geq \frac{\alpha}{2}) + m(h(F_{n_k}, F) \geq \frac{\alpha}{2}) \right] = 0, \quad \forall \alpha > 0. \quad \square$$

Proposition 5.5 *Let $F : \Omega \to ck(X)$ be a measurable multifunction and let $f : \Omega \to X$ be a Σ-measurable function; then the multifunction $\Gamma_f : \Omega \to ck(X)$ defined by:*

$$\Gamma_f(\omega) = \{x \in F(\omega) : \|f(\omega) - x\| = r_\omega\}$$

where $r_\omega = d(f(\omega), F(\omega))$, is Σ-measurable.

Proof: let $\{\phi_n\}$ be a Castaing representation for F, we observe that:

$$r(\omega) = d(f(\omega), F(\omega)) = \inf_{t \in F(\omega)} \|f(\omega) - t\| = \inf_n \|f(\omega) - \phi_n(\omega)\|.$$

Since for every $n \in I\!N$ $\|f(\omega) - \phi_n(\omega)\|$ is Σ-measurable then $d(f, F)$ is Σ-measurable too.
Let $G(\omega) = f(\omega) + r(\omega)X_1$ and let $\{u_n\}_n$ be a dense sequence of X_1. $\{f(\omega) + r(\omega)u_n, n \in I\!N\}$ is a Castaing representation for G and then, by [7] Remark pag. 67, G is measurable.
Since $\Gamma_f(\omega) = F(\omega) \cap G(\omega)$ we obtain that Γ_f takes values in $ck(X)$ and moreover, by Proposition 11.5.6 of [18], Γ_f is measurable. $\square$

Theorem 5.6 *Let $F : \Omega \to ck(X)$ be a totally measurable, multifunction such that $S^1_F \neq \emptyset$. Then, for every $E \in \Sigma$,*

$$(A) - \int_E F dm = (A) - \int_{\tau(E)} \overline{F} d\overline{m}.$$

Proof: by Theorem 5.1

$$(A) - \int_E F dm \subset (A) - \int_{\tau(E)} \overline{F} d\overline{m}.$$

We now prove the converse inclusion. Let $\phi \in S^1_{\overline{F}}$, and $\alpha > 0$ be fixed. By Theorem 5.4 there exists a Σ-measurable $f \in L^1(\Omega, \Sigma, m, X)$ such that $\overline{f} = \phi$ and such that $f \in S^1_{F+\alpha X_1}$.
Let Γ_f be as in Proposition 5.5; since $\Gamma_f(\omega) \subset F(\omega)$ then, by Theorem 4.3, Γ_f admits m-integrable selections.
If $g \in S^1_{\Gamma_f}$, then $g \in S^1_F$ and moreover m-a.e. $\|f(\omega) - g(\omega)\| = d(f(\omega), F(\omega)) \leq \alpha$.
So $f - g$ is a null function and for every $E \in \Sigma$, $\int_E f dm = \int_E g dm$. Since $g \in S^1_F$ then:

$$\int_{\tau(E)} \phi d\overline{m} = \int_E f dm = \int_E g dm \in (A) - \int_E F dm.$$

Then we can conclude with our main result, that is:

Theorem 5.7 *Let m be a bounded finitely additive measure with $L^1(\Omega, \Sigma, m, X)$ complete. If $F : \Omega \to ck(X)$ is a totally measurable multifunction such that $S^1_F \neq \emptyset$ then, for every $E \in \Sigma$,*

$$(A) - \int_E F dm = (B) - \int_E F dm.$$

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