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ON THE PRODUCT OF M-MEASURES IN l-GROUPS

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ABSTRACT. Some extension-type theorems and compactness properties for the product of l-group-valued M-measures are proved.

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1. INTRODUCTION

In the study of probability theory, in many applications it is advisable to deal with set functions, which are not necessarily additive, but satisfy other properties: for example, continuity from below and from above for sequences of sets and "compatibility" with respect to the operations of finite suprema and infima. These functions are called *M-measures* (see [7, 12, 21]).

For example, in decision making, this is the case of the theory of intuitionistic fuzzy events (shortly IF-events), which are pairs $A = (\mu_A, \nu_A)$ of measurable functions $\mu_A, \nu_A : \Omega \to [0, 1]$ such that $\mu_A + \nu_A \leq 1$. For a literature about IF-sets, see [1, 2, 5, 6, 12, 13, 17, 18, 19]. Another application is the theory of joint random variables: in this context the M-measure extension theorem plays a crucial role in the construction of joint observables. Moreover, to consider lattice-group or Riesz space-valued set functions allows to get applications in stochastic processes and in probabilities depending on the time and/or on the informations of the individual.

In this paper we continue the investigation dealt with in [4] and, starting from an extension-type existence theorem for M-measures with values in l-groups, we obtain existence results in the countably compact case for M-measures and product of M-measures.

2. PRELIMINARIES AND BASIC RESULTS

We begin with the following

Definition 2.1. An l-group (lattice ordered group) R is said to be

- **(2.1.1):** *Dedekind complete* if every nonempty subset of R, bounded from above, has supremum in R;
- **(2.1.2):** *super Dedekind complete*, if it is Dedekind complete and for any nonempty set $A \subseteq R$, bounded from above, there exists a countable subset $A^* \subseteq A$, having the same supremum as A.
- **(2.1.3):** A bounded double sequence $(a_{i,j})_{i,j}$ in R is called *regulator* or *(D)-sequence* if, for each $i \in \mathbb{N}$, $a_{i,j} \searrow 0$, that is $a_{i,j} \geq a_{i,j+1}$ for all $j \in \mathbb{N}$ and $\bigwedge_{j \in \mathbb{N}} a_{i,j} = 0$.
- **(2.1.4):** Given a sequence $(r_n)_n$ in R, we say that $(r_n)_n$ *(D)-converges* to an element $r \in R$ if there is a regulator $(a_{i,j})_{i,j}$, such that to every map $\varphi \in \mathbb{N}^{\mathbb{N}}$ there corresponds a positive integer k with

$$|r_n - r| \leq \bigvee_{i=1}^{\infty} a_{i,\varphi(i)}$$

for all $n \geq k$. In this case, we write $(D) \lim_n r_n = r$ or simply $\lim_n r_n = r$, since no confusion can arise.

Definition 2.2. We say that R is *weakly σ-distributive* if, for every (D)-sequence $(a_{i,j})_{i,j}$,

$$\bigwedge_{\varphi \in \mathbb{N}^{\mathbb{N}}} \left(\bigvee_{i=1}^{\infty} a_{i,\varphi(i)} \right) = 0.$$

Remark 2.3. Observe that in weakly σ-distributive l-groups (D)-convergence for sequences coincides with order convergence. An example of a super Dedekind complete weakly σ-distributive l-group is the space $L^0(Y, \Sigma, \nu)$, where (Y, Σ, ν) is a measure space with ν σ-additive and σ-finite, see [14].

From now on, let X be a set and $\mathcal{W}$ be an algebra of subsets of X.

Definition 2.4. A family $\mathcal{A}$ of subsets of X is called *monotone class* if the following properties hold:

(a): $\bigcup_{n\in\mathbb{N}} A_n \in \mathcal{A}$ for every non-decreasing sequence $(A_n)_{n\in\mathbb{N}}$ in $\mathcal{A}$,

(b): $\bigcap_{n\in\mathbb{N}} A_n \in \mathcal{A}$ for every non-increasing sequence $(A_n)_{n\in\mathbb{N}}$ in $\mathcal{A}$.

Definition 2.5. A family $\mathcal{K}$ of subsets of X is called *countably compact class*, if for every sequence $(C_n)_n$ of elements of $\mathcal{K}$ we get $\bigcap_{i\in\mathbb{N}} C_i \neq \emptyset$ whenever $\bigcap_{i=1}^{n} C_i \neq \emptyset$ for any $n \in \mathbb{N}$.

Definition 2.6. A set function $\lambda : \mathcal{W} \to R$ is said to be *countably compact*, if there is a countably compact class $\mathcal{K}$ with the property that for any $A \in \mathcal{W}$ there corresponds a (D)-sequence $(a_{i,j})_{i,j}$ such that to any $\varphi \in \mathbb{N}^{\mathbb{N}}$ two sets $B \in \mathcal{W}$, $C \in \mathcal{K}$ can be found, with $B \subseteq C \subseteq A$ and $\lambda(A \setminus B) \leq \bigvee_{i=1}^{\infty} a_{i,\varphi(i)}$.

In the sequel we will use the following fundamental results ([8, 9, 10, 20]).

Lemma 2.7. ([20], Theorem 3.2.3) *Let $\{(a_{i,j}^{(n)})_{i,j} : n \in \mathbb{N}\}$ be any countable family of regulators. Then for each fixed element $u \in R$, $u \geq 0$, there exists a regulator $(a_{i,j})_{i,j}$ such that, for every $\varphi \in \mathbb{N}^{\mathbb{N}}$,*

$$u \wedge \sum_{n=1}^{\infty} \left(\bigvee_{i=1}^{\infty} a_{i,\varphi(i+n)}^{(n)} \right) \leq \bigvee_{i=1}^{\infty} a_{i,\varphi(i)}.$$

Theorem 2.8. ([11], Theorem 1.6.B) *If $\mathcal{A} \subseteq \mathcal{P}(X)$ is a monotone class of sets such that*

(c): $\mathcal{W} \subseteq \mathcal{A}$,

then $\mathcal{A}$ includes the σ-algebra of subsets of X $\sigma(\mathcal{W})$ generated by $\mathcal{W}$.

Lemma 2.9. ([10], Lemma 413R) *Let $\mathcal{K}$ be a countably compact class of sets. Then there is a countably compact class $\mathcal{K}^* \supseteq \mathcal{K}$ such that $K \cup L \in \mathcal{K}^*$ and $\bigcap_{n\in\mathbb{N}} K_n \in \mathcal{K}^*$ whenever $K, L \in \mathcal{K}^*$ and $(K_n)_n$ is a sequence in $\mathcal{K}^*$.*

Definition 2.10. A set function $\mu : \mathcal{W} \to R$ is called *M-measure* if it satisfies the following properties:

(2.10.i): $\mu(\emptyset) = 0$;

(2.10.ii): $\mu(A \cup B) = \mu(A) \vee \mu(B) = \sup(\mu(A), \mu(B))$ for all $A, B \in \mathcal{W}$;

(2.10.iii): $\mu(A \cap B) = \mu(A) \wedge \mu(B) = \inf(\mu(A), \mu(B))$ for any A, $B \in \mathcal{W}$;

(2.10.iv): μ is *continuous* both from below and from above, that is: if $A_n \nearrow A$, (resp. $B_n \searrow B$), A_n, A $(B_n,\ B) \in \mathcal{W}$, $n \in \mathbb{N}$, then $\mu(A_n) \nearrow \mu(A)$ $(\mu(B_n) \searrow \mu(B))$.

It is known that

Theorem 2.11. ([4], Theorem 3.1) *Let R be a super Dedekind complete weakly σ-distributive l-group. For every bounded R-valued M-measure μ, defined on a ring $\mathcal{W}$, there is a unique M-measure $\overline{\mu}$ defined on the σ-ring $\sigma(\mathcal{W})$ generated by $\mathcal{W}$, extending μ.*

The line of the proof of this theorem is the following: at the first step, the M-measure μ is extended to $\mathcal{W}^+$, which is the class of all sets A of the type

$$A := \cup_{n=1}^{\infty} A_n, \quad \text{with } A_n \subset A_{n+1}, \quad A_n \in \mathcal{W} \quad \text{for all } n \in \mathbb{N},$$

by setting $\mu^+(A) = \lim_n \mu(A_n)$ (the limit exists in R and does not depend on the choice of the sequence $(A_n)_n$).

Successively, it is extended to $\mathcal{W}^*$, which is the ideal generated by $\mathcal{W}^+$, by setting $\mu^*(A) = \inf\{\mu^+(B) : B \in \mathcal{W}^+, B \supset A\}$ for every $A \in \mathcal{W}^*$. Then it is proved that $\overline{\mu} := \mu^*_{|\sigma(\mathcal{W})}$ is an M-measure, extending μ. Finally, in order to prove uniqueness, observe that, if $\nu : \sigma(\mathcal{W}) \to R$ is an M-measure with extends μ, then it coincides with $\overline{\mu}$ on a monotone family containing $\mathcal{W}$, and this concludes the assertion.

We denote by $\overline{\mu} : \sigma(\mathcal{W}) \to R$ this extension. Observe that:

Theorem 2.12. *If μ is countably compact, then $\overline{\mu}$ is countably compact too.*

Proof: Let $\mathcal{K}$ be the countably compact class related with countable compactness of μ and $\mathcal{K}^* \supseteq \mathcal{K}$ be a countably compact class associated with $\mathcal{K}$, closed with respect to finite unions and countable intersections, existing by virtue of Lemma 2.9. Set

$$\begin{aligned} (2.1) \quad \mathcal{L} \;=\; & \{A \in \sigma(\mathcal{W}) : \text{ there is a regulator } (a^{(A)}_{i,j})_{i,j} \text{ such that } \forall\, \varphi \in \mathbb{N}^{\mathbb{N}} \\ & \exists\, B_A \in \sigma(\mathcal{W}),\, C_A \in \mathcal{K}^* : B_A \subseteq C_A \subseteq A \text{ and } \overline{\mu}(A \setminus B_A) \le \bigvee_{i=1}^{\infty} a^{(A)}_{i,\varphi(i)}\}. \end{aligned}$$

It is enough to prove that $\mathcal{L}$ satisfies Theorem 2.8. First of all, the inclusion $\mathcal{W} \subseteq \mathcal{L}$ follows directly from the definition of countably compact measure. We now prove that $\mathcal{L}$ is a monotone class.

If $(A_n)_n$ is a non-decreasing sequence in $\mathcal{L}$, let $A = \bigcup_{n\in\mathbb{N}} A_n$. Since $\overline{\mu}$ is continuous from below, there is a regulator $(b_{i,j})_{i,j}$ with the property that to every $\varphi \in \mathbb{N}^{\mathbb{N}}$ there corresponds a positive integer k with $\overline{\mu}(A \setminus A_n) \le \bigvee_{i=1}^{\infty} b_{i,\varphi(i)}$ whenever $n \ge k$. By (2.1), in correspondence with A_k let $(a^{(k)}_{i,j})_{i,j}$ be the associated regulator, $B_k \in \sigma(\mathcal{W})$, $C_k \in \mathcal{K}^*$ be such that $B_k \subseteq C_k \subseteq A_k \subseteq A$ and

$$\overline{\mu}(A_k \setminus B_k) \le \bigvee_{i=1}^{\infty} a^{(k)}_{i,\varphi(i)}.$$

Set $c^{(k)}_{i,j} = 2(a^{(k)}_{i,j} + b_{i,j})$, $i, j \in \mathbb{N}$. The double sequence $(c_{i,j})_{i,j}$ is a regulator. We obtain:

$$\overline{\mu}(A \setminus B_k) \;\le\; \overline{\mu}(A \setminus A_k) + \overline{\mu}(A_k \setminus B_k) \le \bigvee_{i=1}^{\infty} c^{(k)}_{i,\varphi(i)};$$

moreover, by Lemma 2.7, there exists a (D)-sequence $(f_{i,j})_{i,j}$ such that

$$\overline{\mu}(A \setminus B_k) \le \overline{\mu}(\Omega) \wedge \sum_{n=1}^{\infty} \left(\bigvee_{i=1}^{\infty} c^{(n)}_{i,\varphi(i+n)} \right) \le \bigvee_{i=1}^{\infty} f_{i,\varphi(i)}.$$

Let now $(A_n)_n$ be a non-increasing sequence in $\mathcal{L}$, and set $A = \cap_{n\in\mathbb{N}} A_n$. In correspondence with A_n, let B_n, C_n, $(a^{(n)}_{i,j})_{i,j}$ be as in (2.1). Set $B = \cap_{n\in\mathbb{N}} B_n$, $C = \cap_{n\in\mathbb{N}} C_n$. Then there is a positive integer p such that

$$\overline{\mu}(A \setminus B) \;\le\; \overline{\mu}(A_p \setminus B_p) \le \bigvee_{i=1}^{\infty} a^{(p)}_{i,\varphi(i+p)}$$

and, by Lemma 2.7, there exists a (D)-sequence $(g_{i,j})_{i,j}$ such that

$$\overline{\mu}(A_p \setminus B_p) \;\le\; \overline{\mu}(\Omega) \wedge \sum_{n=1}^{\infty} \left(\bigvee_{i=1}^{\infty} a^{(n)}_{i,\varphi(i+n)} \right) \le \bigvee_{i=1}^{\infty} g_{i,\varphi(i)}.$$

Moreover $B \subseteq C \subseteq A$, $B \in \sigma(\mathcal{W})$ and $C \in \mathcal{K}^*$, since $\mathcal{K}^*$ is closed with respect to countable intersections. This concludes the proof. □

3. EXISTENCE OF PRODUCT MEASURES

Let R be a super Dedekind complete weakly σ-distributive l-group, $(X, \mathcal{S}, \mu)$, $(Y, \mathcal{T}, \nu)$ be two measure spaces, where $\mathcal{S}$, $\mathcal{T}$ are algebras and μ, ν are R-valued countably compact M-measures. We want to define the product measure of μ and ν. By Theorem 2.12 there exist two countably compact M-measures $\overline{\mu} : \sigma(\mathcal{S}) \to R$, $\overline{\nu} : \sigma(\mathcal{T}) \to R$, extending μ and ν respectively.

Let now $\mathcal{E}$ be the family of the *elementary sets*, of the type

$$\bigcup_{l=1}^{n} (A_l \times B_l), \tag{3.1}$$

where $n \in \mathbb{N}$, $A_l \in \sigma(\mathcal{S})$, $B_l \in \sigma(\mathcal{T})$, $l = 1, \dots, n$, and $(A_l \times B_l) \cap (A_s \times B_s) = \emptyset$ whenever $l \neq s$. We prove the following:

Theorem 3.1. *Let μ, ν be R-valued countably compact M-measures as above. Then there is exactly one countably compact M-measure $\overline{\kappa} : \sigma(\mathcal{E}) \to R$ with*

$$\overline{\kappa}(A \times B) = \overline{\mu}(A) \wedge \overline{\nu}(B) \quad \text{for all } A \in \sigma(\mathcal{S}), B \in \sigma(\mathcal{T}). \tag{3.2}$$

Proof: Set

$$\kappa(A \times B) = \overline{\mu}(A) \wedge \overline{\nu}(B), \quad A \in \sigma(\mathcal{S}),\ B \in \sigma(\mathcal{T}).$$

Let now $E = A \times B, F = C \times D \in \sigma(\mathcal{S}) \times \sigma(\mathcal{T})$. We get:

$$\begin{aligned}
\kappa(E \cup F) &= \kappa\left((A \times B) \cup (C \times D)\right) \\
&= \kappa[((A \setminus C) \times B) \cup ((A \cap C) \times (B \cup D)) \cup ((C \setminus A) \times D)] \\
&= (\overline{\mu}(A \setminus C) \wedge \overline{\nu}(B)) \vee (\overline{\mu}(A \cap C) \wedge \overline{\nu}(B \cup D)) \vee (\overline{\mu}(C \setminus A) \wedge \overline{\nu}(D)) \\
&= (\overline{\mu}(A \setminus C) \wedge \overline{\nu}(B)) \vee (\overline{\mu}(A \cap C) \wedge \overline{\nu}(B)) \vee \\
&\vee\ (\overline{\mu}(A \cap C) \wedge \overline{\nu}(D)) \vee (\overline{\mu}(C \setminus A) \wedge \overline{\nu}(D)) \\
&= (\overline{\mu}(A) \wedge \overline{\nu}(B)) \vee (\overline{\mu}(C) \wedge \overline{\nu}(D)) = \kappa(E) \vee \kappa(F).
\end{aligned}$$

Analogously, it is possible to check that

$$\kappa(E \cap F) = \kappa(E) \wedge \kappa(F) \quad \text{for all } E, F \in \sigma(\mathcal{S}) \times \sigma(\mathcal{T}).$$

If $E, F \in \mathcal{E}$, the analogous results follow by virtue of the distributive laws. So, $\kappa : \mathcal{E} \to R$ can be defined as follows:

$$\kappa\left(\bigcup_{l=1}^{n}(A_l \times B_l)\right) = \bigvee_{l=1}^{n}(\overline{\mu}(A_l) \wedge \overline{\nu}(B_l)).$$

Now, in order to prove that κ is countably compact, we show that for each $A \in \mathcal{E}$ a (D)-sequence $(c_{i,j})_{i,j}$ can be found, with the property that:

$$\forall \varphi \in \mathbb{N}^{\mathbb{N}}, \exists B \in \mathcal{E}, \exists C \in \mathcal{K} \quad \text{with } B \subseteq C \subseteq A \text{ and } \kappa(A \setminus B) \leq \bigvee_{i=1}^{\infty} c_{i,\varphi(i)}. \tag{3.3}$$

As $\overline{\mu}$ and $\overline{\nu}$ are countably compact measures, there are countably compact classes $\mathcal{K}_1 \subseteq \mathcal{P}(X)$, $\mathcal{K}_2 \subseteq \mathcal{P}(Y)$, with the following property: for all $A \in \sigma(\mathcal{S})$ and $B \in \sigma(\mathcal{T})$ there is a regulator

$(\alpha_{i,j})_{i,j}$ such that $\forall\,\varphi\in\mathbb{N}^{\mathbb{N}}\ \exists\ E\in\sigma(\mathcal{S})$, $C\in\mathcal{K}_1$, $F\in\sigma(\mathcal{T})$, $D\in\mathcal{K}_2$, with $E\subseteq C\subseteq A$, $F\subseteq D\subseteq B$,

$$\sup\{\overline{\mu}(A\setminus E),\overline{\nu}(B\setminus F)\}\leq\bigvee_{i=1}^{\infty}\alpha_{i,\varphi(i)}.$$

Set now

$$\mathcal{H}=\{E=C\times D:\ C\in\mathcal{K}_1,\ D\in\mathcal{K}_2\}\,.$$

It is easy to see that $\mathcal{H}$ is countably compact. By [10], Lemma 451H and Lemma 2.9 there exists a countably compact class $\mathcal{K}$ containing $\mathcal{H}$, and closed with respect to finite unions and countable intersections.

Let A be any element of $\mathcal{E}$. There exist $n\in\mathbb{N}$, $A_l\in\mathcal{S}$, $B_l\in\mathcal{T}$, $l=1,\ldots,n$ such that $A=\cup_{l=1}^{n}(A_l\times B_l)$. Since $\overline{\mu}$, $\overline{\nu}$ are countably compact M-measures, then to $l=1,\ldots,n$ there correspond two (D)-sequences $(a^{(l)}_{i,j})_{i,j}$, $(b^{(l)}_{i,j})_{i,j}$ and $H_l\in\mathcal{K}_1$, $G_l\in\mathcal{K}_2$, $E_l,F_l\in\mathcal{S}$ with $E_l\subseteq H_l\subseteq A_l$, $F_l\subseteq G_l\subseteq B_l$,

$$\overline{\mu}(A_l\setminus E_l)\leq\bigvee_{i=1}^{\infty}a^{(l)}_{i,\varphi(i+l)},$$

$$\overline{\nu}(B_l\setminus F_l)\leq\bigvee_{i=1}^{\infty}b^{(l)}_{i,\varphi(i+l)}.$$

By Lemma 2.7, there are two regulators $(a_{i,j})_{i,j}$, $(b_{i,j})_{i,j}$, with

$$[\overline{\mu}(X)]\wedge\sum_{l=1}^{\infty}\left(\bigvee_{i=1}^{\infty}a^{(l)}_{i,\varphi(i+l)}\right)\leq\bigvee_{i=1}^{\infty}a_{i,\varphi(i)},$$

$$[\overline{\nu}(Y)]\wedge\sum_{l=1}^{\infty}\left(\bigvee_{i=1}^{\infty}b^{(l)}_{i,\varphi(i+l)}\right)\leq\bigvee_{i=1}^{\infty}b_{i,\varphi(i)}\quad\text{for all }\varphi\in\mathbb{N}^{\mathbb{N}}.$$

Put $c_{i,j}=a_{i,j}\vee b_{i,j}$, $i,j\in\mathbb{N}$, and

$$C=\bigcup_{l=1}^{n}(H_l\times G_l),\qquad B=\bigcup_{l=1}^{n}(E_l\times F_l):$$

we get $B\subseteq C\subseteq A$, $C\in\mathcal{K}$, and

$$\begin{aligned}\kappa(A\setminus B)&=\kappa\left(\left[\bigcup_{l=1}^{n}(A_l\times B_l)\right]\setminus\left[\bigcup_{s=1}^{n}(E_s\times F_s)\right]\right)\\&=\kappa\left(\left[\bigcup_{l=1}^{n}((A_l\setminus E_l)\times B_l)\right]\cup\left[\bigcup_{s=1}^{n}(A_s\times(B_s\setminus F_s))\right]\right)\\&=\left[\bigvee_{l=1}^{n}(\overline{\mu}(A_l\setminus E_l)\wedge\overline{\nu}(B_l))\right]\vee\left[\bigvee_{s=1}^{n}(\overline{\mu}(A_s)\wedge\overline{\nu}(B_s\setminus F_s))\right]\\&\leq\left[\bigvee_{l=1}^{n}\overline{\mu}(A_l\setminus E_l)\right]\vee\left[\bigvee_{s=1}^{n}\overline{\nu}(B_s\setminus F_s)\right]\leq\bigvee_{i=1}^{\infty}c_{i,\varphi(i)}.\end{aligned}$$

Countable compactness of κ follows.

We now prove **(2.10.iv)**, namely that $\kappa(A_n)\searrow 0$ (resp. $\kappa(B_n)\nearrow\kappa(B)$) whenever $(A_n)_n$ is a non-increasing sequence in $\mathcal{E}$, with $\bigcap_{n\in\mathbb{N}}A_n=\emptyset$ (resp. $B_n\in\mathcal{E}$, $n\in\mathbb{N}$, $B_n\nearrow B$, $B\in\mathcal{E}$).

Pick now arbitrarily any sequence $(A_n)_n$ in $\mathcal{E}$, with $A_n \searrow \emptyset$. Since κ is countably compact, then in correspondence with each positive integer n a (D)-sequence $(d_{i,j}^{(n)})_{i,j}$ and two elements $B_n \in \mathcal{E}$, $C_n \in \mathcal{K}$ can be found, with $B_n \subseteq C_n \subseteq A_n$ and

$$\kappa(A_n \setminus B_n) \leq \bigvee_{i=1}^{\infty} d_{i,\varphi(i+n)}^{(n)}.$$

Again by Lemma 2.7, there exists a (D)-sequence $(d_{i,j})_{i,j}$ with the property that

$$[\kappa(X \times Y)] \wedge \sum_{n=1}^{\infty} \left(\bigvee_{i=1}^{\infty} d_{i,\varphi(i+n)}^{(n)} \right) \leq \bigvee_{i=1}^{\infty} d_{i,\varphi(i)}.$$

Set now $D_n = \bigcap_{l=1}^{n} C_l$. We get

$$\bigcap_{n \in \mathbb{N}} D_n \subseteq \bigcap_{n \in \mathbb{N}} A_n = \emptyset.$$

Since $\mathcal{K}$ is a countably compact class, a positive integer m can be found, with

$$\bigcap_{l=1}^{m} B_l \subseteq D_m = \bigcap_{l=1}^{m} C_l = \emptyset.$$

For each $n \geq m$ we obtain

$$\begin{aligned} \kappa(A_n) &\leq \kappa(A_m) = \kappa \left(A_m \setminus \left[\bigcap_{l=1}^{m} B_l \right] \right) = \kappa \left(\bigcup_{l=1}^{m} (A_m \setminus B_l) \right) \\ &\leq \kappa \left(\bigcup_{l=1}^{m} (A_l \setminus B_l) \right) = \bigvee_{l=1}^{m} \kappa(A_l \setminus B_l) \leq \bigvee_{i=1}^{\infty} d_{i,\varphi(i)} \end{aligned}$$

and then $\lim_n \kappa(A_n) = 0$.

Let now $B_n \in \mathcal{E}$ $(n \in \mathbb{N})$, $B_n \nearrow B$, $B \in \mathcal{E}$. Then $B \setminus B_n \searrow \emptyset$, and hence

$$\begin{aligned} \kappa(B) &= \kappa((B \setminus B_n) \cup B_n) = \kappa(B \setminus B_n) \vee \kappa(B_n) \\ &\leq \kappa(B \setminus B_n) \vee \left(\bigvee_{i=1}^{\infty} \kappa(B_i) \right). \end{aligned}$$

Thus we get:

$$\kappa(B) \leq \lim_n \kappa(B \setminus B_n) \vee \left(\bigvee_{i=1}^{\infty} \kappa(B_i) \right) = \bigvee_{i=1}^{\infty} \kappa(B_i) \leq \kappa(B),$$

and hence $\kappa(B) = \lim_i \kappa(B_i)$. Furthermore, if $C_n \searrow C$, then $C_n \setminus C \searrow \emptyset$,

$$\kappa(C_n) = \kappa((C_n \setminus C) \cup C) = \kappa(C_n \setminus C) \vee \kappa(C),$$

and

$$\bigwedge_{n=1}^{\infty} \kappa(C_n) = \left(\bigwedge_{n=1}^{\infty} \kappa(C_n \setminus C) \right) \vee \kappa(C) = 0 \vee \kappa(C) = \kappa(C).$$

Thus we proved that κ is an R-valued countably compact M-measure, defined on $\mathcal{E}$. By Theorem 2.12 there is a (unique) countably compact M-measure $\overline{\kappa}$, defined on $\sigma(\mathcal{E})$ and extending κ. □

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