Stieltjes-type integrals for metric semigroup-valued functions defined on unbounded intervals

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Abstract

We introduce the $GH_k$ integral for functions defined on (possibly) unbounded subintervals of the extended real line and with values in metric semigroups. Basic properties and convergence theorems for this integral are deduced.

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Stieltjes-type integrals are widely studied in the literature: for example, meaningful results can be found in [8, 9, 10, 23]. In particular, in [13, 14, 15] and in a more abstract setting in [8, 9], an integral \((\text{GH}_k\text{-integral})\) for real-valued functions defined in a compact subinterval of the real line has been investigated, which generalizes the integral studied by Š. Schwabik in [24]: the latter includes also the classical Kurzweil-Henstock and Henstock-Stieltjes integrals. Some examples of other particular cases of the \(\text{GH}_k\) integral are illustrated in [8, 9].

In this paper we extend the \(\text{GH}_k\) integral to the case of metric semigroup-valued functions, defined on (possibly) unbounded subintervals of the extended real line, and we prove some convergence theorems. Similar results were proved in [5] in the context of the Kurzweil-Henstock integral, for which the \(\text{GH}_k\) integral is substantially a particular case; moreover, in this paper we prove also an extension Cauchy-type theorem.

For a literature existing on the Kurzweil-Henstock integral in the context of metric semigroups, we refer to [5, 16, 26] and their bibliography, while for Riesz-space valued functions we recall [1, 2, 3, 4, 17, 18, 19, 20, 21, 22]. A particular example of metric semigroup is the set \(L(\mathbb{R})\) of fuzzy numbers (see also Section 2 and [5]).

2 Metric semigroups.

**Definition 2.1.** A metric semigroup is a structure \((X, \rho, +, \cdot)\), where \(\rho : X \times X \rightarrow \mathbb{R}, + : X \times X \rightarrow X, \cdot : \mathbb{R} \times X \rightarrow X\) satisfy the following conditions:

(i) \((X, \rho)\) is a complete metric space;

(ii) \((X, +)\) is a commutative semigroup endowed with a neutral element 0;

(iii) \(\rho(w + y, z + t) \leq \rho(w, z) + \rho(y, t)\) for any \(w, y, z, t \in X\);

(iv) \(\rho(\alpha w, \alpha y) \leq |\alpha| \rho(w, y)\) for all \(\alpha \in \mathbb{R}\) and \(w, y \in X\);

(v) \(\alpha(w + y) = \alpha w + \alpha y\) for each \(\alpha \in \mathbb{R}, w, y \in X\);

(vi) \((\alpha + \beta)w = \alpha w + \beta w\) for every \(\alpha, \beta \in \mathbb{R}_0^+, w \in X, 0 \cdot w = 0\) and \(1 \cdot w = w\) for each \(w \in X\).

A metric semigroup \((X, \rho, +, \cdot)\) is called invariant, if

\[\rho(w + z, y + z) = \rho(w, y)\]

for any \(w, y, z \in X\).

Observe that a consequence of invariance and the triangular property is the following condition, which will be useful in the sequel:
Stieltjes-type integrals for metric semigroup-valued functions

(vii) \( \rho(w + y, z) \leq \rho(w, t) + \rho(y + t, z) \) whenever \( x, y, z, t \in X \).

An example of metric semigroup is the set of all fuzzy numbers (see also [5, 26]).

**Definition 2.2.** A fuzzy number is a function \( \mu : \mathbb{R} \to [0, 1] \) satisfying the following conditions:

(j) there exists \( x_0 \in \mathbb{R} \) such that \( \mu(x_0) = 1 \);

(jj) the \( \alpha \)-cut set \( \mu_\alpha = \{ x \in \mathbb{R} : \mu(x) \geq \alpha \} \) is convex for \( \alpha \in [0, 1] \);

(jjj) \( \mu \) is upper semi-continuous, i.e. any \( \alpha \)-cut \( \mu_\alpha \) is a closed subset of \( \mathbb{R} \);

(jv) the support \( \{ x \in \mathbb{R} : \mu(x) > 0 \} \) of the function \( \mu \) is a compact set.

Any real number \( u_0 \) can be identified with a fuzzy number \( \mu_0 \) in the following way:

\[ \mu_0(x) = \chi\{u_0\}(x), \]

i.e. \( \mu_0(u_0) = 1 \), and \( \mu_0(x) = 0 \) if \( x \neq u_0 \).

The set of all fuzzy numbers is denoted by \( L(\mathbb{R}) \).

We now endow \( L(\mathbb{R}) \) with a metric and a linear structure (see also [5, 26]).

We define the Hausdorff distance \( H \) on the set of all compact possibly degenerate intervals in \( \mathbb{R} \):

\[ H([a, b], [c, d]) = \max(|c - a|, |d - b|). \]

Let \( \mu, \nu \in L(\mathbb{R}) \). It is easy to check that, for every \( \alpha \in (0, 1] \), there exist \( a, b, c, d \in \mathbb{R} \) (depending on \( \alpha \)) such that \( \mu_\alpha = [a, b], \nu_\alpha = [c, d] \). So, for \( \mu, \nu \in L(\mathbb{R}) \), set

\[ \rho(\mu, \nu) = \sup\{H(\mu_\alpha, \nu_\alpha) : \alpha \in (0, 1]\}. \]

Using this definition, \( (L(\mathbb{R}), \rho) \) becomes a complete metric space.

To define a linear structure on \( L(\mathbb{R}) \), recall that every fuzzy number is completely determined by its \( \alpha \)-cuts. Hence, for any \( \mu, \nu \in L(\mathbb{R}), \alpha \in \mathbb{R}^+ \) and \( \lambda \in \mathbb{R} \), set

\[ (\mu + \nu)_\alpha = \mu_\alpha + \nu_\alpha, \]

\[ (\lambda \mu)_\alpha = \lambda \mu_\alpha \]

(here, \( V + Z = \{ v + z : v \in V, z \in Z \}; \lambda V = \{ \lambda v : v \in V \} \)).

Finally, we note that \( (L(\mathbb{R}), +) \) is not a group, but only a semigroup (see also [5]), in fact let \( \mu \in L(\mathbb{R}) \) be defined by the formula:

\[ \mu(x) = \begin{cases} x, & \text{if } x \in [0, 1]; \\ 2 - x, & \text{if } x \in [1, 2]; \\ 0, & \text{otherwise}. \end{cases} \]
Then \(-\mu = (-1) \cdot \mu\) is given by

\[
-\mu(x) = \begin{cases} 
-x, & \text{if } x \in [-1, 0]; \\
2 + x, & \text{if } x \in [-2, -1]; \\
0, & \text{otherwise.}
\end{cases}
\]

Note that \(\mu(x) + (-\mu(x))\) is not the zero element \(0 \equiv \chi_{\{0\}}(x)\), but

\[
\mu(x) + (-\mu(x)) = \begin{cases} 
1 - \frac{x}{2}, & \text{if } x \in [0, 2]; \\
1 + \frac{x}{2}, & \text{if } x \in [-2, 0]; \\
0, & \text{otherwise.}
\end{cases}
\]

On the other hand the subset \(R_0 \subset L(\mathbb{R})\) consisting of all functions \(\chi_{\{a\}}, a \in \mathbb{R}\), is group isomorphic to the commutative group \((\mathbb{R}, +)\).

3 The construction of the integral.

From now on we denote by capital letters the elements of the extended real line and by small letters the real numbers. Let \([A, B]\) be a (possibly unbounded) interval of the extended real line, and \(\mathcal{F}\) be the family of all closed convex subsets. By partition (or \(k\)-partition) of a set \(W \in \mathcal{F}\) we denote a finite collection

\[
\Pi = \{(\xi_1; F_{1,1}, \ldots, F_{1,k}), \ldots, (\xi_q; F_{q,1}, \ldots, F_{q,k})\} = \{(\xi_1; E_1), \ldots, (\xi_q; E_q)\} \quad (1)
\]

such that

(i) \(F_{i,j} \in \mathcal{F}\) for all \(i = 1, \ldots, q\) and \(j = 1, \ldots, k\);

(ii) \(\bigcup_{j=1}^{k} F_{i,j} = E_i\) for all \(i = 1, \ldots, q\);

(iii) \(\bigcup_{i=1}^{q} E_i = W\);

(iv) \(\xi_i \in E_i (i = 1, \ldots, q)\);

(v) the \(F_{i,j}\)'s are pairwise non-overlapping;

(vi) \(\sup F_{i,j} = \inf F_{i,j+1}\) whenever \(i = 1, \ldots, q\) and \(j = 1, \ldots, k - 1\).

A finite collection \(\Pi\) as in (1), satisfying conditions (i), (ii), (iv), (v) and (vi), but not necessarily (iii), is said to be a decomposition (or \(k\)-decomposition) of \(W\).
**Definitions 3.1.**

- A gauge is a map $\gamma$ defined in $[A, B]$ and taking values in the set of all open intervals in $\tilde{\mathbb{R}}$, such that $\xi \in \gamma(\xi)$ for every $\xi \in [A, B]$ and $\gamma(\xi)$ is a bounded open interval (with respect to the topology of $[A, B]$) for every $\xi \in \mathbb{R} \cap [A, B]$.

- Given a gauge $\gamma$, a $k$-decomposition of $[A, B]$ of the type
  \[ \Pi = \{(\xi_i; E_i), i = 1, \ldots, q\} \]
  is said to be $\gamma$-fine if $\xi_i \in E_i \subset \gamma(\xi_i)$ for all $i = 1, \ldots, q$. Observe that for any gauge $\gamma$ there always exists a $\gamma$-fine $k$-partition (see also [8, 11]).

- Given $[a, b] \subset \mathbb{R}$ and a map $\delta : [a, b] \to \mathbb{R}^+$, a partition $\Pi$ of $[a, b]$ as in (2) is said to be $\delta$-fine if $\xi_i \in E_i \subset (\xi_i - \delta(\xi_i), \xi_i + \delta(\xi_i))$ for all $i = 1, \ldots, q$. In any case we note that, if $E_i$ is an unbounded interval, then the element $\xi_i$ associated with $E_i$ is necessarily $+\infty$ or $-\infty$: otherwise $\gamma(\xi_i)$ should be a bounded interval and contain an unbounded interval, a contradiction.

From now on, we assume that $X$ is an invariant metric semigroup. Given any $k$-decomposition $\Pi$ as in (1) and a function $U : [A, B] \times F^k \to X$, we call Riemann sum of $U$ (and we write $\sum_{\Pi} U$) the expression
\[ \sum_{i=1}^{q} U(\xi_i; F_{i,1}, \ldots, F_{i,k}). \]

We now introduce the $GH_k$ integral for $X$-valued functions defined on $[A, B] \times F^k$. We will show that this concept can be formulated equivalently both with gauges and with positive maps $\delta$.

**Definition 3.2.** We say that a function $U : [A, B] \times F^k \to X$ is $GH_k$ integrable on $[A, B]$ if there exists $I \in X$ such that for all $\epsilon > 0$ there correspond a function $\delta : [A, B] \to \mathbb{R}^+$ and a positive real number $P$ such that
\[ \rho\left(I, \sum_{\Pi} U\right) \leq \epsilon \]
whenever $\Pi$ is a $\delta$-fine $k$-partition of any bounded interval $[a, b]$ with $[a, b] \supset [A, B] \cap [-P, P]$. In this case we say that $I$ is the $GH_k$ integral of $U$, and we denote the element $I$ by the symbol $(GH_k) \int_{A}^{B} U$, writing usually $U \in GH_k[A, B]$.

Analogously it is possible to define the integral $(GH_k) \int_{c}^{d} U$ for each subinterval $[c, d] \subset [A, B]$.

**Remark 3.3.** We note that the $GH_k$ integral is well-defined, that is there exists at most one element $I$, satisfying condition (4) (see also [5]).
We now give the following characterization of $GH_k$ integrability.

**Theorem 3.4.** A function $U : [A, B] \times F^k \to X$ is $GH_k$ integrable if and only if there is $J \in X$ such that for all $\varepsilon > 0$ there exists a gauge $\gamma$ such that

$$\rho(J, \sum_{\Pi} U) \leq \varepsilon \quad (5)$$

whenever $\Pi$ is a $\gamma$-fine partition of $[A, B]$, and in this case we have $\int_{A}^{B} f = J$.

**Proof:** See also [3], Theorem 3.3., and [5]. $\square$

4 Elementary properties of the $GH_k$ integral

The proof of the following proposition is similar to the corresponding one in [5].

**Proposition 4.1.** If $U_1, U_2 \in GH_k[A, B]$ and $c_1, c_2 \in \mathbb{R}$, then $c_1 U_1 + c_2 U_2 \in GH_k[A, B]$, and

$$(GH_k) \int_{A}^{B} (c_1 U_1 + c_2 U_2) = c_1 (GH_k) \int_{A}^{B} U_1 + c_2 (GH_k) \int_{A}^{B} U_2.$$

(Here we intend by $-U$ the entity $(-1) \cdot U$)

**Theorem 4.2.** A map $U : [A, B] \times F^k \to X$ is $GH_k$ integrable if and only if for all $\varepsilon > 0$ there exists a gauge $\gamma = \gamma(\varepsilon)$ on $[A, B]$ such that

$$\rho \left( \sum_{\Pi} U, \sum_{\Pi'} U \right) \leq \varepsilon \quad (6)$$

whenever $\Pi, \Pi'$ are $\gamma$-fine $k$-partitions of $[A, B]$.

**Proof:** We follow the lines of the proof of Proposition 3.5 of [5].

The necessary part is straightforward.

We now turn to the sufficient part. Let $U$ satisfy (6), and set $\varepsilon = 1/n$, with $n \in \mathbb{N}$. Then for all $n$ there exists a gauge $\gamma_n$ on $[A, B]$ such that

$$\rho \left( \sum_{\Pi} U, \sum_{\Pi'} U \right) \leq \frac{1}{n} \quad (6)$$

whenever $\Pi_1, \Pi_2$ are $\gamma_n$-fine partitions of $[A, B]$. Put $\eta_n = \gamma_1 \cap \gamma_2 \cap \ldots \cap \gamma_n$ for all $n \in \mathbb{N}$, and set

$$A_n = \{x \in X : \exists \eta_n \text{-fine partition } \Pi_1 : x = \sum_{\Pi_1} U \}, \quad n \in \mathbb{N}.$$
If \( x, y \in A_n \), then \( \rho(x, y) \leq 1/n \), and hence

\[
\text{diam } A_n = \text{diam } A_n \leq \frac{1}{n}.
\]

Since \( \eta_{n+1} \subset \eta_n \), we obtain \( A_{n+1} \subset A_n \). Since \( X \) is complete, there exists exactly one element \( I \in \cap_{n=1}^{\infty} A_n \).

Pick arbitrarily \( \varepsilon > 0 \), and choose \( n \in \mathbb{N} \) such that \( \frac{1}{n} < \varepsilon \). If \( \Pi \) is any \( \eta_n \)-fine partition, then

\[
\sum_{\Pi} U \in A_n.
\]

Since \( I \in A_n \), we obtain

\[
\rho(I, \sum_{\Pi} U) \leq \frac{1}{n} < \varepsilon.
\]

Therefore \( U \) is \( GH_k \)-integrable on \([A, B]\) and \( I = \int_A^B U \). \( \square\)

We now investigate \( GH_k \)-integrability on subintervals, by proceeding similarly as in [8].

**Proposition 4.3.** If \( U \in GH_k[A, B] \), then \( U \in GH_k[c, d] \) for each \( [c, d] \subset [A, B] \), and

\[
(GH_k) \int_A^B U = (GH_k) \int_A^c U + (GH_k) \int_c^B U
\]

whenever \( A < c < B \).

**Proof:** We begin with the first statement. Without loss of generality, we can assume that \( [c, d] = [A, d] \), with \( A < d < B \). Let \( \gamma \) be any gauge on \([A, B]\), pick any two \( \gamma \)-fine \( k \)-partitions \( \Pi_1, \Pi_2 \) of \([A, d]\), and let \( \Pi' \) be a \( \gamma \)-fine \( k \)-partition of \([d, B]\). Such a partition does exist, by virtue of the Cousin lemma. Then, for \( j = 1, 2 \), \( \Pi_j := \Pi' \cup \Pi_j \) is a \( \gamma \)-fine partition of \([A, B]\). Since

\[
\rho \left( \sum_{\Pi_1} U, \sum_{\Pi_2} U \right) = \rho \left( \sum_{\Pi_1} U, \sum_{\Pi_2} U \right),
\]

then the assertion follows from the Cauchy criterion.

We now turn to the last part. For every \( \varepsilon > 0 \) there exists a gauge \( \gamma \) such that for each \( \gamma \)-fine \( k \)-partition \( \Pi_1 \) of \([A, c]\) and \( \Pi_2 \) of \([c, B]\) we get

\[
\rho \left( \sum_{\Pi_1} U, (GH_k) \int_A^c U \right) \leq \varepsilon,
\]

and

\[
\rho \left( \sum_{\Pi_2} U, (GH_k) \int_c^B U \right) \leq \varepsilon.
\]

Hence, if \( \Pi = \Pi_1 \cup \Pi_2 \), we have also

\[
\rho \left( \sum_{\Pi} U, (GH_k) \int_A^B U \right) \leq \varepsilon.
\]
We obtain:

\[ 0 \leq \rho \left( (GH_k) \int_A^c U + (GH_k) \int_c^B U \right) \]

\[ \leq \rho \left( \sum_{i=1}^{l_1} U_i (GH_k) \int_A^c U \right) + \rho \left( \sum_{i=1}^{l_2} U_i (GH_k) \int_c^B U \right) + \rho \left( \sum U_i (GH_k) \int_B^A U \right) \]

\[ \leq 3 \varepsilon. \]

By arbitrariness of \( \varepsilon \in \mathbb{R}^+ \) we get that

\[ (GH_k) \int_B^A U = (GH_k) \int_A^c U + (GH_k) \int_c^B U. \]

This completes the proof. \( \Box \)

In order to establish a converse of the previous result, we now introduce the following property.

**Definition 4.4.** Let \( U : [A, B] \times \mathcal{F}^k \to X \) and fix a point \( x_0 \in [A, B] \). We say that \( U \) satisfies condition \([H1]) at \( x_0 \) if for all \( \varepsilon > 0 \) there exists a positive real number \( \eta = \eta(\varepsilon; x_0) \) such that

\[ \rho \left( U(x_0; [w_0, w_1], \ldots, [w_{k-1}, w_k]) - U(x_0; [w_0, w_1], \ldots, [w_{k-1}, w_k]) \right) \leq \varepsilon \]

whenever \( \bigcup_{l=0}^{2} \left( \bigcup_{i=1}^{k} [w_{i-1}, w_i] \right) \subset [x_0 - \eta, x_0 + \eta] \) and \( w_0 = w_1 = \ldots = w_k \).

Note that \( H1) \) is a kind of “quasi-additivity” of the set function \( U \). In many cases, when \( X = \mathbb{R} \), \( U \) is defined by means of suitable "differences" (for example, \( U(t; [u, v]) = V(t; v) - V(t; u) \) when \( k = 1 \) or

\[ U(t; [w_0, w_1], \ldots, [w_{k-1}, w_k]) = V(t; w_1, \ldots, w_k) - V(t; w_0, \ldots, w_{k-1}) \]

for \( k \geq 2 \); then, if \( k = 1 \), property \( H1) \) is automatically satisfied (see also [24], Theorem 1.11, pp. 10-12); while for \( k \geq 2 \) it is implied by the condition of "existence of the iterated limit \( J \)" used by A. G. Das and S. Kundu (see [8], Definition 2.9., p. 69).

We now prove the following result on additivity.

**Theorem 4.5.** Let \( U : [A, B] \times \mathcal{F}^k \to X \) satisfy condition \( H1) \) at \( c \in [A, B] \). If \( U \in GH_k[A, c] \) and \( U \in GH_k[c, B] \), then \( U \in GH_k[A, B] \) and

\[ (GH_k) \int_A^B U = (GH_k) \int_A^c U + (GH_k) \int_c^B U. \]
Proof: By hypothesis, for every $\varepsilon > 0$ there exist a function $\delta^* : [A, B] \to \mathbb{R}^+$ and a positive real number $P$ (without loss of generality, greater than $|c|$) with the following property: for all $\delta^*$-fine $k$-partitions $\Pi_1$ of any bounded interval $[a_1, b_1] \subset [A, c]$, $[a_1, b_1] \supset [A, c] \cap [-P, P]$ and $\Pi_2$ of every bounded interval $[a_2, b_2] \subset [c, B]$, $[a_2, b_2] \supset [c, B] \cap [-P, P]$ we get
\[
\rho \left( \sum_{\Pi_1} U, (GH_k) \int_A^c U \right) \leq \varepsilon, \quad \rho \left( \sum_{\Pi_2} U, (GH_k) \int_c^B U \right) \leq \varepsilon.
\]
Let $\eta = \eta(\varepsilon; c)$ be related to condition H1 at $c$, and set $\delta(x) = \min\{\delta^*(x), |x - c|\}$ if $x \in [A, B] \setminus \{c\}$, $\delta(c) = \min\{\delta^*(c), \eta\}$. Pick now any bounded interval $[a, b] \subset [A, B]$, $[a, b] \supset [A, B] \cap [-P, P]$, and any $\delta^*$-fine $k$-partition
\[
\Pi = \{(\xi_i; F_{i,1}, \ldots, F_{i,k}), i = 1, \ldots, q\}
\]
of $[a, b]$. There exists $m$ with $1 \leq m \leq q$, such that $c = \xi_m$ and $\bigcup_{j=1}^{k} F_{i,j}$ contains $c$ if and only if $i = m$ (see also [8, 24]). We get:
\[
\sum_{\Pi} U = \sum_{i=1}^{m-1} U(\xi_i; F_{i,1}, \ldots, F_{i,k}) + U(c; F_{m,1}, \ldots, F_{m,k}) + \sum_{i=m+1}^{q} U(\xi_i; F_{i,1}, \ldots, F_{i,k}).
\]
Consider now the points
\[
c - \delta(c) < x_{m-1,k} = y_{m,0} < \ldots < y_{m,k} = c = z_{m,0} < \ldots < z_{m,k} = x_{m+1,0} < c + \delta(c).
\]
The parts of the partition $\Pi$ for $i = 1, \ldots, m-1$ ($i = m+1, \ldots, q$) and the single family $\{(c; [y_{m,0}, y_{m,1}], \ldots, [y_{m,k-1}, y_{m,k}])\}$, $\{(c; [z_{m,0}, z_{m,1}], \ldots, [z_{m,k-1}, z_{m,k}])\}$ form a $\delta^*$-fine $k$-partition $\Pi_1$ ($\Pi_2$) of $[a, c]$ ($[c, b]$). So, we have:
\[
\rho \left( \sum_{\Pi} U, (GH_k) \int_A^c U + (GH_k) \int_c^B U \right) \\
\leq \rho \left( \sum_{\Pi_1} U, (GH_k) \int_A^c U \right) + \rho \left( \sum_{\Pi_2} U, (GH_k) \int_c^B U \right) + \rho \left( \sum_{\Pi_1} U, \frac{\sum_{\Pi_2} U}{\sum_{\Pi_2} U} \sum_{\Pi_2} U \right) \\
\leq 2\varepsilon + \rho(U(c; F_{m,1}, \ldots, F_{m,k}), U(c; [y_{m,0}, y_{m,1}], \ldots, [y_{m,k-1}, y_{m,k}])) \\
+ U(c; [z_{m,0}, z_{m,1}], \ldots, [z_{m,k-1}, z_{m,k}]) \leq 3\varepsilon.
\]
From this it follows that $U \in GH_k[A, B]$ and
\[
(GH_k) \int_A^B U = (GH_k) \int_A^c U + (GH_k) \int_c^B U.
\]
This concludes the proof. □
5 Convergence theorems

We begin with a version of the Saks-Henstock lemma (see also [5], Proposition 4.1). Here, the symbol $|\cdot|$ denotes the Lebesgue measure.

Lemma 5.1. Let $U : [A, B] \times F^k \to X$ be $GH_k$ integrable on $[A, B]$. Then for every $\varepsilon > 0$ there exists a gauge $\gamma$ on $[A, B]$ such that, for every $\gamma$-fine $k$-decomposition of $[A, B]$,

$$\Pi = \{(t_i; F_{i,1}, \ldots, F_{i,k}), i = 1, \ldots, m\} = \{(t_i; E_i), i = 1, \ldots, m\}, \quad (7)$$

where $\bigcup_{j=1}^k F_{i,j} = E_i$, $i = 1, \ldots, m$, we have

$$\rho\left(\sum_{i=1,\ldots,m,|E_i|<+\infty} U(t_i; F_{i,1}, \ldots, F_{i,k}), \sum_{i=1}^m (GH_k) \int_{E_i} U\right) \leq \varepsilon.$$

Proof: (see also [5]) Choose arbitrarily $\varepsilon > 0$, and let $\gamma$ be a gauge on $[A, B]$ existing in correspondence with $\varepsilon$, according to Theorem 3.4. Fix arbitrarily any $\gamma$-fine $k$-decomposition $\Pi$ of $[A, B]$ as in (7), and let $\text{int} E_i$ be the interior of $E_i$, $i = 1, \ldots, m$. Since the $E_i$’s are non-overlapping, the set $[A, B] \setminus \bigcup_{i=1}^m (\text{int} E_i)$ is empty or is the union of non-overlapping (possibly bounded or not) intervals $B_1, \ldots, B_p$. Let $\eta > 0$. Since $U$ is $GH_k$ integrable on each $B_j$, for each $j = 1, \ldots, p$ there exists a gauge $\gamma_j$ on $B_j$ such that $\gamma_j(x) \subset \gamma(x)$ for all $x \in B_j$ and

$$\rho\left(\sum_{i=1}^m U_i (GH_k) \int_{B_j} U\right) < \frac{\eta}{p+1}$$

for every $\gamma_j$-fine partition $\Pi_j$ of $B_j$. Let now $\Pi_j$ be such a partition. We observe that

$$\Pi := \{(t_i; F_{i,1}, \ldots, F_{i,k}), i = 1, \ldots, m\} \cup \bigcup_{j=1}^p \Pi_j$$
is a \( \gamma \)-fine partition of \([A, B]\). Then we have:

\[
\rho \left( \sum_{i=1}^{m} U(t_i; F_{i,1}, \ldots, F_{i,k}) \sum_{i=1}^{m} (GH_k) \int_{E_i} U \right) \\
= \rho \left( \sum_{i=1}^{m} U(t_i; F_{i,1}, \ldots, F_{i,k}) + \sum_{j=1}^{p} \sum_{\Lambda_j} U, \sum_{i=1}^{m} (GH_k) \int_{E_i} U + \sum_{j=1}^{p} \sum_{\Lambda_j} U \right) \\
\leq \rho \left( \sum_{\Pi \in A} U, (GH_k) \int_{A} U \right) \\
+ \rho \left( \sum_{j=1}^{p} (GH_k) \int_{B_j} U, \sum_{j=1}^{p} \sum_{\Pi_j} U \right) \\
\leq \varepsilon + \rho \left( (GH_k) \int_{B_j} U, \sum_{j=1}^{p} \sum_{\Pi_j} U \right) \\
< \varepsilon + \sum_{j=1}^{p} \eta < \varepsilon + \eta.
\]

Since the inequality

\[
\rho \left( \sum_{i=1}^{m} U(t_i; F_{i,1}, \ldots, F_{i,k}) \sum_{i=1}^{m} (GH_k) \int_{E_i} U \right) < \varepsilon + \eta
\]

holds for any \( \eta > 0 \), then the assertion follows by arbitrariness of \( \eta \). \( \square \)

We now prove a version of a Hake’s type theorem, which is an extension of the Cauchy theorem. To do this, let \( U : [A, B] \times \mathcal{F}^k \to X \) be with \( U \in GH_k[A, c] \) for all \( c \in [A, B] \), fix \( I \in X \) and let us introduce the following condition:

- **H2)** for every \( \varepsilon > 0 \) there exists a left neighborhood \( U \) of \( B \) such that

\[
\rho \left( I, (GH_k) \int_{A} U + U(B; F_1, \ldots, F_k) \right) \leq \varepsilon
\]

whenever \( F_1, \ldots, F_k \in \mathcal{F} \) are pairwise non-overlapping and such that \( U \ni c \leq \inf F_1 \leq \sup F_j = \inf F_{j+1}, \, j = 1, \ldots, k - 1, \) and \( \sup F_k = B \).

In the literature several situations are considered, when, in the Riemann sums, only the terms where the involved intervals are bounded are taken: this can be done simply by postulating it or by requiring the condition

\[
U(\pm \infty; \Lambda_1, \ldots, \Lambda_k) = 0
\]

for every choice of \( \Lambda_j \in \mathcal{F}, \, j = 1, \ldots, k \).
Observe that, when \( B = +\infty \) and we require (8), \( H2) \) can be automatically replaced by the simpler condition of existence in \( X \) of the limit

\[
\lim_{c \to B^-} (GH_k) \int_A^c U.
\]  

Finally, we note that, when \( X = \mathbb{R} \), property \( H2) \) is implied by the two conditions of existence in \( \mathbb{R} \) of the limit as in (9) and of “existence of the iterated limit (from the left) \( J^- \)” used by A. G. Das and S. Kundu (see [8]) when \( k \geq 2 \).

For \( k = 1 \), \( H2) \) is equivalent to the existence in \( \mathbb{R} \) of the limit in [24], formula (1.11), p. 15.

**Theorem 5.2.** Let \( A \in \mathbb{R^+} \), \( U : [A, B] \times \mathcal{F}_k \to X \) be such that \( U \in GH_k[A, c] \) for every \( c \in [A, B] \), and suppose that there is an element \( I \in X \) such that \( H2) \) holds.

Then \( U \in GH_k[A, B] \) and \( (GH_k) \int_A^B U = I \).

Moreover, if \( U \in GH_k[A, B] \), then \( \lim_{c \to B^-} (GH_k) \int_A^c U = (GH_k) \int_A^B U \) (this last result is independent on \( H2) \).

**Proof:** Let \( (c_p)_p \) be a strictly increasing sequence in \([A, b[ \) with \( c_p \uparrow B \) and \( c_0 = A \). For every \( p \in \mathbb{N} \) and \( \varepsilon > 0 \) there exists a gauge \( \gamma_p : [A, c_p] \to \mathbb{R^+} \), such that

\[
\rho \left( \sum_{i=1}^{n} U, (GH_k) \int_{A}^{c_p} U \right) \leq \frac{\varepsilon}{2^p}
\]

whenever \( \Pi_p \) is any \( \gamma_p \)-fine \( k \)-partition of \([A, c_p] \).

For every \( \xi \in [A, B] \) there exists exactly one \( p = p(\xi) \in \mathbb{N} \) such that \( \xi \in [c_{p(\xi)} - 1, c_{p(\xi)}] \). Given \( \xi \in [A, B] \), choose \( \tilde{\gamma}(\xi) > 0 \) such that \( \tilde{\gamma}(\xi) < \gamma_{p(\xi)}(\xi) \) and \( \tilde{\gamma}(\xi) \cap [A, B] \subseteq [A, c_{p(\xi)}(\xi)] \). Let \( c \in [A, B] \) and

\[
\tilde{\Pi} := \{ (\xi_i, F_{i,1}, \ldots, F_{i,k}), i = 1, \ldots, n \} = \{ (\xi_i, E_i), i = 1, \ldots, n \},
\]

with \( \bigcup_{j=1}^{k} F_{i,j} = E_i, \ i = 1, \ldots, n \), be a \( \tilde{\gamma} \)-fine \( k \)-partition of \([A, c] \). For every \( i = 1, \ldots, n \) we get:

\[
E_i \subseteq \tilde{\gamma}(\xi_i) \subseteq [A, c_{p(\xi_i)}(\xi_i)].
\]

Furthermore, \( E_i \subseteq \gamma_{p(\xi_i)}(\xi_i) \). For every \( p \in \mathbb{N} \), let us indicate by

\[
\sum_{i=1}^{n} \rho \left( U(\xi_i; F_{i,1}, \ldots, F_{i,k}), (GH_k) \int_{E_i} U \right)
\]

the sum of those terms of

\[
\sum_{i=1}^{n} \rho \left( U(\xi_i; F_{i,1}, \ldots, F_{i,k}), (GH_k) \int_{E_i} U \right)
\]
for which \( \xi_i \in [c_{p-1}, c_p] \). By Lemma 5.1 we obtain

\[
\rho \left( \sum_{i=1}^{n} U(\xi_i; F_{i,1}, \ldots, F_{i,k}), (GH_k) \int_{E_i} \right) \leq \frac{\varepsilon}{2^p}
\]

for all \( p \in \mathbb{N} \). Since \( U \in GH_k[A, c] \) for every \( c \in [A, B] \), then by Proposition 4.3 we have

\[
(GH_k) \int_{A}^{c} U = \sum_{i=1}^{n} (GH_k) \int_{E_i} U.
\]

So we get:

\[
\rho \left( \sum_{i=1}^{n} U(\xi_i; F_{i,1}, \ldots, F_{i,k}), (GH_k) \int_{A}^{c} U \right) = \rho \left( \sum_{i=1}^{n} U(\xi_i; F_{i,1}, \ldots, F_{i,k}), \sum_{i=1}^{n} (GH_k) \int_{E_i} U \right) \\
\leq \sum_{p=1}^{\infty} \rho \left( \sum_{i=1}^{n} U(\xi_i; F_{i,1}, \ldots, F_{i,k}), \sum_{i=1}^{n} (GH_k) \int_{E_i} U \right) \\
\leq \sum_{p=1}^{\infty} \frac{\varepsilon}{2^p} = \varepsilon.
\]

Let \( U \) be related with condition \( \text{H2} \), and pick a gauge \( \gamma \) on \([A, B]\) such that \( \gamma(\xi) \subset \hat{\gamma}(\xi) \) if \( \xi \in [A, B] \), and \( \gamma(B) \subset U \). Let

\[
\Pi := \{(\xi_i; F_{i,1}, \ldots, F_{i,k}), i = 1, \ldots, n\} = \{(\xi_i; E_i), i = 1, \ldots, n\}
\]

be any arbitrary \( \gamma \)-fine \( k \)-partition of \([A, B]\), where \( \bigcup_{j=1}^{k} F_{i,j} = E_i \) and \( E_i = [x_{i-1,k}, x_{i,k}], i = 1, \ldots, n \): we get \( x_{n,k} = B \) and hence \( \xi_n = B \) (if not, then \( E_n \subset \hat{\gamma}(\xi_n) \subset [A, c_{p(\xi_n)}] \) and thus \( x_{n,k} < B \), a contradiction). We have, thanks to the condition formulated in the hypothesis and using property \( \text{(vii)} \) of the function \( \rho \),

\[
\rho \left( \left.I \sum_{1 \leq i \leq k} U \right|_{\Pi} \right) \leq \rho \left( I, \sum_{i=1}^{n-1} U(\xi_i; F_{i,1}, \ldots, F_{i,k}) + U(B; F_{n,1}, \ldots, F_{n,k}) \right) \\
\leq \rho \left( \sum_{i=1}^{n-1} U(\xi_i; F_{i,1}, \ldots, F_{i,k}), (GH_k) \int_{A}^{x_{n-1,k}} U \right) \\
+ \rho \left( I, (GH_k) \int_{A}^{x_{n-1,k}} U + U(B; F_{n,1}, \ldots, F_{n,k}) \right) \\
\leq \rho \left( \sum_{i=1}^{n-1} U(\xi_i; F_{i,1}, \ldots, F_{i,k}), (GH_k) \int_{A}^{x_{n-1,k}} U \right) + \varepsilon.
\]
As \( x_{n-1,k} < B \) and \( \{ (\xi_i; F_{i,1}, \ldots, F_{i,k}), i = 1, \ldots, n-1 \} \) is a \( \hat{\gamma} \)-fine \( k \)-partition of \([A, x_{n-1,k}]\), we get
\[
\rho \left( \sum_{i=1}^{n-1} U(\xi_i; F_{i,1}, \ldots, F_{i,k}), (GH_k) \int_{A}^{x_{n-1,k}} U \right) \leq \varepsilon,
\]
and hence
\[
\rho \left( I, \sum_{\Pi} U \right) \leq 2\varepsilon.
\]
From this the assertion of the first part of the theorem follows.

We now turn to the last part. Since, by hypothesis, \( U : [A, B] \times F^k \to X \) is \( GH_k \) integrable on \([A, B]\), then \( U \) is \( GH_k \) integrable on \([A, c]\) for every \( A < c \leq B \). So for all \( \varepsilon > 0 \) and \( c \in [A, B] \) there exists \( \delta_1 : [A, c] \to \mathbb{R}^+ \) such that for every \( \delta_1 \)-fine \( k \)-partition \( \Pi' \) of \([A, c]\) we get:
\[
\rho \left( \sum_{\Pi'} U, (GH_k) \int_{A}^{c} U \right) \leq \varepsilon.
\]
Moreover, by \( GH_k \) integrability on \([A, B]\) (see also Definition 3.2), for any \( \varepsilon > 0 \) there exist \( \delta : [A, B] \to \mathbb{R}^+ \) and \( P \in ]A, B[ \) such that for every bounded interval \([d_1, d_2] \subset [A, B]\) with \([d_1, d_2] \supset [P, P]\) and for each \( \delta \)-fine \( k \)-partition \( \Pi \) of \([d_1, d_2]\) we have
\[
\rho \left( \sum_{\Pi} U, (GH_k) \int_{A}^{B} U \right) \leq \varepsilon.
\]
Let now \( \varepsilon > 0, c > P, \delta_2(x) := \min\{\delta(x), \delta_1(x)\}, x \in [A, c], \) and \( \Pi \) be any \( \delta_2 \)-fine \( k \)-partition of \([A, c]\). Then we get:
\[
\rho \left( (GH_k) \int_{A}^{c} U, (GH_k) \int_{A}^{B} U \right) \leq \rho \left( \sum_{\Pi} U, (GH_k) \int_{A}^{c} U \right) + \rho \left( \sum_{\Pi} U, (GH_k) \int_{A}^{B} U \right) \leq 2\varepsilon.
\]
Thus the theorem is completely proved. \( \Box \)

**Remark 5.3.** An analogous version of Theorem 5.2 holds, if we consider, in our "limit operations" and calculus, the point \( A \) from the right instead of the point \( B \) from the left.

This concept will be useful in the sequel.

**Definition 5.4.** A sequence of integrable functions \( (U_h : [A, B] \times F^k \to X)_{h} \) is said to be **equiintegrable** if for any \( \varepsilon > 0 \) there exists a gauge \( \gamma \) on \([A, B]\) such that
\[
\rho \left( \sum_{\Pi} U_h, (GH_k) \int_{A}^{B} U_h \right) \leq \varepsilon
\]
for any \( \gamma \)-fine partition \( \Pi \) and every \( h \in \mathbb{N} \).
We now prove the following convergence theorems for the $GH_k$ integral in the context of metric semigroups.

**Theorem 5.5.** Let $(U_h)_h$ be an equiintegrable sequence and let

$$\lim_{h \to +\infty} \rho(U_h(t; \Lambda_1, \ldots, \Lambda_k), U(t; \Lambda_1, \ldots, \Lambda_k)) = 0$$

for any $t \in [A, B]$ and uniformly with respect to $\Lambda_1, \ldots, \Lambda_k \in \mathcal{F}$. Then $U$ is $GH_k$ integrable on $[A, B]$, and

$$\lim_{h \to +\infty} \rho \left( (GH_k) \int_A^B U_h, (GH_k) \int_A^B U \right) = 0.$$

**Proof:** First of all, we observe that for each $\varepsilon > 0$, there exist: a non-negative function $E : [A, B] \times \mathcal{F}^k \to \mathbb{R}$, strictly positive on $([A, B] \cap \mathbb{R}) \times \mathcal{F}^k$, $GH_k$ integrable in $[A, B]$, with

$$(GH_k) \int_A^B E \leq \frac{\varepsilon}{2}$$

(for example,

$$E(t; \Lambda_1, \ldots, \Lambda_k) = \sum_{j=1}^k |\Lambda_j| \frac{\varepsilon}{2\pi(1 + t^2)}, \quad t \in [A, B],$$

with the convention $E(\pm \infty; \Lambda_1, \ldots, \Lambda_k) = 0$ for every choice of $\Lambda_j \in \mathcal{F}$, $j = 1, \ldots, k$); a gauge $\gamma_0$ on $[A, B]$, such that

$$\sum_{i=1, \ldots, n, |I_i| < +\infty} \mathcal{E}(t_i; F_{i,1}, \ldots, F_{i,k}) \leq \varepsilon$$

(11)

for each $\gamma_0$-fine partition $\Pi$ of $[A, B]$,

$$\Pi := \{(t_i; F_{i,1}, \ldots, F_{i,k}), i = 1, \ldots, n\} = \{(t_i; I_i), i = 1, \ldots, n\},$$

with $\bigcup_{j=1}^k F_{i,j} = I_i$, $i = 1, \ldots, n$.

Let now $\varepsilon > 0$, $\gamma$ be as in 5.4, $\hat{\gamma} = \gamma \cap \gamma_0$, and

$$\Pi := \{(t_i; F_{i,1}, \ldots, F_{i,k}), i = 1, \ldots, n\} = \{(t_i; I_i), i = 1, \ldots, n\},$$

be any $\hat{\gamma}$-fine $k$-partition of $[A, B]$, where $\bigcup_{j=1}^k F_{i,j} = I_i$, $i = 1, \ldots, n$. Then for each $i = 1, \ldots, n$ there exists a positive integer $h_i$ such that

$$\rho(U_h(t_i; F_{i,1}, \ldots, F_{i,k}), U(t_i; F_{i,1}, \ldots, F_{i,k})) \leq \mathcal{E}(t_i; F_{i,1}, \ldots, F_{i,k})$$

(12)
whenever \( h \geq h_i \). Pick now \( h \geq \max_{i=1,\ldots,n} h_i \). From (11) and (12) we have:

\[
\rho \left( \sum_{\Pi} U_h, \sum_{\Pi} U \right) = \rho \left( \sum_{i=1,\ldots,n} U_h(t_i; F_i,1,\ldots,F_i,k), \sum_{i=1,\ldots,n} U(t_i; F_i,1,\ldots,F_i,k) \right) \leq \rho(U_h(t_i; F_i,1,\ldots,F_i,k), U(t_i; F_i,1,\ldots,F_i,k)) \leq \varepsilon.
\]

It follows that

\[
\lim_{h \to +\infty} \rho \left( \sum_{\Pi} U_h, \sum_{\Pi} U \right) = 0.
\]

Now we get:

\[
\rho \left( \sum_{\Pi} U_i (GH_k) \int_A^B U_h \right) \leq \rho \left( \sum_{\Pi} U_i \sum_{\Pi} U_h \right) + \rho \left( \sum_{\Pi} U_h, (GH_k) \int_A^B U_h \right) \leq 2\varepsilon.
\]

Choose now arbitrarily two \( \tilde{\gamma} \)-fine partitions \( \Pi \) and \( \Pi' \) of \([A,B]\), and let \( h^* = \max\{h_i, h_i', h_i''\} \), where the integers \( h_i, h_i', h_i'' \) associated to \( \Pi \) and \( \Pi' \) respectively have the same role as the \( h_i \)’s in (12). We get:

\[
\rho \left( \sum_{\Pi} U_i (GH_k) \int_A^B U_{h^*} \right) \leq \rho \left( \sum_{\Pi} U_i, (GH_k) \int_A^B U_{h^*} \right) \leq 4\varepsilon.
\]

Integrability of \( U \) on \([A,B]\) follows from (13) and the Cauchy criterion 4.2.

Finally, to every \( \varepsilon > 0 \) there corresponds a gauge \( \gamma \) on \([A,B]\) such that for any \( \gamma \)-fine \( k \)-partition \( \Pi \) there exists \( h \in \mathbb{N} \) with

\[
\rho \left( (GH_k) \int_A^B U_h, (GH_k) \int_A^B U \right) \leq \rho \left( (GH_k) \int_A^B U_h, \sum_{\Pi} U_h \right) \leq 3\varepsilon
\]

for all \( h \geq \gamma \). This implies that

\[
\lim_{h \to +\infty} \rho \left( (GH_k) \int_A^B U_h, (GH_k) \int_A^B U \right) = 0. \quad \Box
\]
The next step is to prove a version of the convergence theorem with respect to the "uniform convergence". To this aim we introduce the following concept.

**Definition 5.6.** Given a sequence of functions \( (U_n : [A, B] \times F^k \to X)_{n \in \mathbb{N} \cup \{0\}} \), we say that the \( U_n \)'s, \( n \geq 1 \), variationally uniformly converge to \( U_0 \) if to every \( \varepsilon > 0 \) an integer \( n_0 \) can be found, such that

\[
\rho \left( \sum_{t=1}^{q} U_n(t; F_{i,1}, \ldots, F_{i,k}), \sum_{t=1}^{q} U_0(t; F_{i,1}, \ldots, F_{i,k}) \right) \leq \varepsilon
\]

for every \( n \geq n_0 \) and any \( k \)-partition \( \Pi = \{(t_i, F_{i,1}, \ldots, F_{i,k}), i = 1, \ldots, q\} \) of \([A, B]\), where \( \bigcup_{j=1}^{k} F_{i,j} = I_i, i = 1, \ldots, q \).

Observe that, if \( k = 1 \) and

\[
U_n(t; [u, v]) = [g(v) - g(u)] \cdot f_n(t), \quad n \in \mathbb{N} \cup \{0\},
\]

where \( g : [A, B] \to \mathbb{R} \) is of bounded variation and the sequence \( (f_n : [A, B] \to X)_{n} \) is uniformly convergent to \( f_0 \) on \([A, B]\), then the \( U_n \)'s variationally uniformly converge to \( U_0 \). In this case, under the hypothesis of uniform convergence of \( (f_n)_{n} \) to \( f_0 \), if the \( f_n \)'s, \( n \geq 1 \), are Henstock-Stieltjes integrable with respect to \( g \), then \( f_0 \) is too, and we get the exchange of limits under the sign of integral.

An example in which this happens if when we take \( X = L(\mathbb{R}) \) (i. e. the set of all fuzzy numbers), and define \( f_n : [0, 1] \to X \) by setting \( f_n(x) = \chi_{[0,1]}[x-1/n, x+1/n] \), \( n \in \mathbb{N} \), then the sequence \( (f_n)_{n} \) is uniformly convergent to the "identity" function (in the sense that the generic element \( x \in [0, 1] \) is identified with the element \( \chi_{\{x\}} \)).

**Theorem 5.7.** Let \( (U_n : [A, B] \times F^k \to X)_{n} \) be a sequence of functions, \( GH_k \) integrable on \([A, B]\) and variationally uniformly convergent to a map \( U \).

Then \( U \) is \( GH_k \) integrable on \([A, B]\) and

\[
\lim_{n \to +\infty} \rho \left( (GH_k) \int_{A}^{B} U_n, (GH_k) \int_{A}^{B} U \right) = 0.
\]

**Proof:** Let \( \varepsilon > 0 \), and take \( n_0 = n_0(\varepsilon) \) according to variationally uniform convergence. Then

\[
\rho \left( \sum_{\Pi_1} U, \sum_{\Pi_2} U \right) \leq \rho \left( \sum_{\Pi_1} U, \sum_{\Pi_1} U_{n_0} \right) + \rho \left( \sum_{\Pi_1} U_{n_0}, \sum_{\Pi_2} U \right) + \rho \left( \sum_{\Pi_2} U_{n_0}, \sum_{\Pi_2} U \right) 
\]

\[
\leq 2\varepsilon + \rho \left( \sum_{\Pi_1} U_{n_0}, \sum_{\Pi_2} U_{n_0} \right)
\]
for any two partitions $\Pi_1, \Pi_2$ of $[A,B]$. Since $U_{n_0}$ is $GH_k$ integrable on $[A,B]$, then there is a map $\delta = \delta_{n_0} : [A,B] \to \mathbb{R}^+$, such that, for any two $\delta$-fine $k$-partitions $\Pi_1, \Pi_2$ of $[A,B]$,\[
\rho \left( \sum_{\Pi_1} U_{n_0}, \sum_{\Pi_2} U_{n_0} \right) \leq \varepsilon ,
\]
and hence\[
\rho \left( \sum_{\Pi_1} U, \sum_{\Pi_2} U \right) \leq 3\varepsilon .
\]
Thus $U$ is $GH_k$ integrable on $[A,B]$, by virtue of the Cauchy criterion 4.2. So there exists a map $\delta' : [A,B] \to \mathbb{R}^+$ such that\[
\rho \left( \sum_{\Pi} U, \int_A^B (GH_k) U \right) \leq \varepsilon
\]
for each $\delta'$-fine partition $\Pi$ of $[A,B]$. Fix $n \geq n_0$ and choose $\kappa_n : [A,B] \to \mathbb{R}^+$ such that\[
\rho \left( \sum_{\Pi} U_n, \int_A^B (GH_k) U_n \right) \leq \varepsilon
\]
whenever $\Pi$ is a $\kappa_n$-fine partition of $[A,B]$. Put $\delta_n = \min\{\delta', \kappa_n\}$: for any $\delta_n$-fine $k$-partition $\Pi$ of $[A,B]$ we obtain\[
\rho \left( \int_A^B (GH_k) U_n, \int_A^B U \right) \leq \rho \left( \int_A^B U_n, \int_A^B U \right) \leq \rho \left( \int_A^B U_n, \sum_{\Pi} U \right) + \rho \left( \sum_{\Pi} U, \sum_{\Pi} U_n \right) + \rho \left( \sum_{\Pi} U_n, \int_A^B U_n \right) \leq 3\varepsilon ,
\]
and thus the last part of the assertion. □

References


Stieltjes-type integrals for metric semigroup-valued functions


