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The Henstock-Kurzweil integral for functions defined on unbounded intervals and with values in Banach spaces

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ABSTRACT. A Henstock-Kurzweil-type integral for functions defined on a (possibly unbounded) subinterval on the extended real line and with values in Banach spaces is investigated.

1 Introduction.

In the literature, there are several studies about the Henstock-Kurzweil integral in Banach spaces: among them, we recall Cao, Fremlin and Mendoza ([2-6]). In this paper we introduce and investigate a Henstock-Kurzweil-type integral for Riesz-space-valued functions defined on (not necessarily bounded) subintervals of the extended real line. We prove some basic properties, among which the fact that our integral contains the generalized Riemann integral and that every simple function which vanishes outside of a set of finite Lebesgue measure is integrable according to our definition, and in this case our integral coincides with the usual one.

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2 The Henstock-Kurzweil integral in Banach spaces.

Let $I\!N$ be the set of all strictly positive integers, $I\!R$ the set of the real numbers, $I\!R^+$ be the set of all strictly positive real numbers, $\widetilde{I\!R}$ the set of all extended real numbers. We will construct a type of integral for Banach-space-valued maps (with respect to the Lebesgue measure defined on intervals, not necessarily bounded), containing the improper Riemann integral. From now on, we denote by $[A,B]$ a closed interval or halfline contained in $\widetilde{I\!R}$, or the whole of $\widetilde{I\!R}$, and by Δ the set of all positive real-valued functions, defined on $[A,B]$. Moreover, given a measurable set $E \subset \widetilde{I\!R}$, we denote by $|E|$ its Lebesgue measure (this quantity can be finite or $+\infty$). Throughout this section, our integral deals with Banach-space-valued functions defined on $[A,B]$, but it can be investigated analogously if we take functions defined on $I\!R$ or on halflines of the type $[a,+\infty)$ or $(-\infty,a]$, with $a \in I\!R$.

Definitions 2.1 A *subpartition* Π of $[A,B]$ is a set of pairs (I_k,ξ_k), $k = 1,\dots,p$, such that $\xi_k \in I_k \quad \forall\, k$, and the I_k's are non-overlapping closed intervals, contained in $[A,B]$. A *partition* $\Pi = \{(I_k,\xi_k),\, k = 1,\dots,p\}$ of $[A,B]$ is a subpartition of $[A,B]$ with $\bigcup_{k=1}^{p} I_k = [A,B]$.

A *gauge* is a map γ defined in $[A,B]$ and taking values in the set of all open intervals in $\widetilde{I\!R}$, such that $\xi \in \gamma(\xi)$ for every $\xi \in [A,B]$ and $\gamma(\xi)$ is a bounded open interval for every $\xi \in I\!R \bigcap [A,B]$. Given a gauge γ, we will say that a partition $\Pi = \{(I_k,\xi_k), k = 1,\dots,p\}$ of $[A,B]$ is *γ-fine* if $I_k \subset \gamma(\xi_k) \quad \forall\, k = 1,\dots,p$. Given a bounded interval $[a,b] \subset I\!R$ and a map $\delta : [a,b] \to I\!R^+$, a partition $\Pi = \{(I_k,\xi_k), k = 1,\dots,p\}$ of $[a,b]$ is said to be *δ-fine* if $I_k \subset (\xi_k - \delta(\xi_k), \xi_k + \delta(\xi_k)) \quad \forall\, k = 1,\dots,p$.

We note that, if I_k is an unbounded interval, then the element ξ_k associated with I_k is necessarily $+\infty$ or $-\infty$: otherwise $\gamma(\xi_k)$ should be a bounded interval and contain an unbounded interval: contradiction.

Let S be any Banach space. Given any partition $\Pi = \{(I_k,\xi_k), k = 1,\dots,p\}$ of $[A,B]$ and a function $f : [A,B] \to S$, we call *Riemann sum* of f (and we write $\sum_{\Pi} f$) the quantity

$$\sum_{k=1}^{p} |I_k|\, f(\xi_k), \tag{1}$$

where in the sum in (1) only the terms for which I_k is a bounded interval are included. This can be required by simply postulating it or by defining the measure of an unbounded interval as $+\infty$, by requiring $f(+\infty) = f(-\infty) = 0$ and by means of the convention $0 \cdot (+\infty) = 0$ (see also [7], p. 65).

We now formulate our definition of Henstock-Kurzweil integral for functions defined on $[A, B]$ and taking values in a Banach space S.

Definition 2.2 We say that a function $f : [A, B] \to S$ is *Henstock-Kurzweil integrable* (in short *HK-integrable*) on $[A, B]$ if there exists an element $I \in S$ such that $\forall\, \varepsilon > 0$ there exist a function $\delta \in \Delta$ and a positive real number P such that

$$\left\| \sum_{\Pi} f - I \right\| \leq \varepsilon \tag{2}$$

whenever $\Pi = \{(I_k, \xi_k), k = 1, \ldots, p\}$ is a δ-fine partition of any bounded interval $[a, b]$ with $[a, b] \supset [A, B] \bigcap [-P, P]$ and $[a, b] \subset [A, B]$. In this case we say that I is the *HK-integral of f,* and we denote the element I by the symbol $\int_A^B f$. Later we will prove that our integral is well-defined, that is such an I is uniquely determined.

We now prove the following characterization of HK-integrability.

Theorem 2.3 *A function $f : [A, B] \to S$ is HK-integrable if and only if there exists $J \in S$ such that $\forall\, \varepsilon > 0$ there exists a gauge γ such that*

$$\left\| \sum_{\Pi} f - J \right\| \leq \varepsilon \tag{3}$$

whenever $\Pi = \{(I_k, \xi_k), k = 1, \ldots, p\}$ is a γ-fine partition of $[A, B]$, and in this case we have $\int_A^B f = J$.

Proof: We begin with the "only if" part. By hypothesis, $\forall\, \varepsilon > 0$ there exist a function $\delta \in \Delta$ and a positive real number P such that (2) holds. We now define on $[A, B]$ a gauge γ in the following way:

$$\gamma(\xi) = \begin{cases} (\xi - \delta(\xi), \xi + \delta(\xi)) & \text{if } \xi \in [A, B] \bigcap I\!R, \\[2ex] [-\infty, -P) & \text{if } \xi = -\infty \text{ and } A = -\infty, \\[2ex] (P, +\infty] & \text{if } \xi = +\infty \text{ and } B = +\infty. \end{cases}$$

We observe that every γ-fine partition $\Pi = \{(I_k, \xi_k), k = 1, \dots, p\}$ of $[A, B]$ is such that $I_k \subset \gamma(\xi_k)\ \forall\, k = 1, \dots, p$. In the case $A = -\infty$, $B = +\infty$, the partition Π contains two unbounded intervals, which we call J and K: of course, if $\inf J = -\infty$ and $\sup K = +\infty$, then the ξ_k's associated with J and K are $-\infty$ and $+\infty$ respectively. Then, since Π is γ-fine, we have $J \subset \gamma(-\infty)$ and $K \subset \gamma(+\infty)$. Then $J \subset [-\infty, -P)$ and $K \subset (P, +\infty]$. So, if $a = \sup J$ and $b = \inf K$, then $[a, b]$ is a bounded interval, containing $[-P, P]$. If Π' is the restriction of Π to $[a, b]$, then Π' is δ-fine, and by construction we get

$$\sum_{\Pi'} f = \sum_{\Pi} f. \tag{4}$$

In this case, the assertion follows from (2) and (4).

In the case $A \in I\!\!R$, $B = +\infty$, the partition Π contains only an unbounded interval K, with $\sup K = +\infty$. Let P be associated with K as above, and $b = \inf K$: we have $P \leq b$. We note that, without loss of generality, P can be taken greater than $|A|$. Thus, $[A, b]$ is a bounded interval, containing $[-P, P]$, and the assertion follows by proceeding as in the previous case. The case $A = -\infty$, $B \in I\!\!R$ is analogous to the previous one. Finally, if $[A, B]$ is bounded, then the assertion is straightforward, because in this case the number P can be taken greater than $\max(|A|, |B|)$ and, of course, (2) holds even in the case $[a, b] = [A, B]$. This concludes the proof of the "only if" part.

We now turn to the "if" part. By hypothesis, we know that $\forall\, \varepsilon > 0$ there exists a gauge γ satisfying (3). By definition of gauge, there exist $\delta_1, \delta_2 \in \Delta$ such that

$$\gamma(\xi) = (\xi - \delta_1(\xi), \xi + \delta_2(\xi)) \quad \forall\, \xi \in [A, B] \bigcap I\!\!R.$$

For such ξ's, let $\delta(\xi) = \min\{\delta_1(\xi), \delta_2(\xi)\}$. Moreover, if $+\infty$ and $-\infty$ belong to $[A, B]$, and $\gamma(-\infty) = [-\infty, P_1^*)$, $\gamma(+\infty) = (P_2^*, +\infty]$, put $P_1 = \min\{P_1^*, -1\}$, $P_2 = \max\{P_2^*, 1\}$, $P = \max\{-P_1, P_2\}$: we note that, in the case $A \in I\!\!R$ (resp. $B \in I\!\!R$), P can be chosen greater than $|A|$ (resp. $|B|$); moreover, set $\delta(-\infty) = \delta(+\infty) = P$. Let now $[a, b] \subset [A, B]$ be any bounded interval, containing $[A, B] \bigcap [-P, P]$, and $\Pi = \{(I_k, \xi_k) : k = 1, \dots, p\}$ be a δ-fine partition of $[a, b]$. Let Π' be that partition of $[A, B]$, whose elements are the ones of Π with the addition of $([A, a], A)$, if $A = -\infty$, and $([b, B], B)$, if $B = +\infty$: we note that Π' is γ-fine. This follows from the fact that, if (I_k, ξ_k) is any element of Π, then

$$I_k \subset (\xi_k - \delta(\xi_k), \xi_k + \delta(\xi_k)) \subset (\xi_k - \delta_1(\xi_k), \xi_k + \delta_2(\xi_k)) = \gamma(\xi_k),$$

and from the following inclusions:

$$(b, +\infty] \subset (P, +\infty] \subset (P_2, +\infty] \subset (P_2^*, +\infty] = \gamma(+\infty),$$

$$[-\infty, a) \subset [-\infty, P) \subset [-\infty, P_1) \subset [-\infty, P_1^*) = \gamma(-\infty).$$

Then, taking into account that the Riemann sum concerning the partition Π' is done without considering the unbounded intervals, we get $\sum_{\Pi'} f = \sum_{\Pi} f$. From this and (3) the assertion follows, by proceeding analogously as at the end of the proof of the converse implication. This concludes the proof of the theorem. □

Remark 2.4 We note that the Henstock-Kurzweil integral is well-defined, that is there exists at most one element I, satisfying condition (3): indeed, if $\exists$ such two elements I, J, then $\forall \varepsilon > 0$ $\exists$ two gauges γ_1, γ_2 such that, for each γ_1-fine partition Π and for every γ_2-fine partition Π' of $[A, B]$ we have

$$\left\| \sum_{\Pi} f - I \right\| \leq \varepsilon$$

and

$$\left\| \sum_{\Pi'} f - J \right\| \leq \varepsilon$$

respectively. Let now $\gamma(\xi) = \gamma_1(\xi) \bigcap \gamma_2(\xi)$, $\forall \xi \in [A, B]$ and take any γ-fine partition Π'': then Π'' is both γ_1- and γ_2-fine, and thus we have

$$0 \leq \|I - J\| \leq 2\,\varepsilon.$$

By arbitrariness of $\varepsilon > 0$, it follows that $\|I - J\| = 0$, and thus $I = J$. So our HK-integral is well-defined. □

We now state the main properties of the HK-integral.

Proposition 2.5 *If f_1, f_2 are HK-integrable on $[A, B]$ and $c_1, c_2 \in I\!R$, then $c_1 f_1 + c_2 f_2$ is HK-integrable on $[A, B]$ and*

$$\int_A^B (c_1 f_1 + c_2 f_2) = c_1 \int_A^B f_1 + c_2 \int_A^B f_2.$$

The proof is similar to the one of [7], Theorems 2.5.1 and 2.5.3.

Proposition 2.6 *Let $A, B \in \widetilde{I\!R}$, and c be such that $A < c < B$. If $f : [A,B] \to S$ is HK-integrable both on $[A,c]$ and on $[c,B]$, then f is HK-integrable on $[A,B]$ and*

$$\int_A^B f = \int_A^c f + \int_c^B f.$$

Proof: In correspondence with HK-integrability of f on $[A,c]$ and $[c,B]$, $\forall\, \varepsilon > 0$ there exist two mappings $\underline{\delta} : [A,c] \to I\!R^+$, $\overline{\delta} : [c,B] \to I\!R^+$, and two positive real numbers $\underline{P}$ and $\overline{P}$ (without loss of generality, $\underline{P} > |c|$, $\overline{P} > |c|$) such that, if $\underline{\Pi}$ is any $\underline{\delta}$-fine partition of any bounded interval $[a_1, b_1] \subset [A,c]$, $[a_1,b_1] \supset [A,c] \bigcap [-\underline{P}, \underline{P}]$ and $\overline{\Pi}$ is any $\overline{\delta}$-fine partition of any bounded interval $[a_2,b_2] \subset [c,B]$, $[a_2,b_2] \supset [c,B] \bigcap [-\overline{P}, \overline{P}]$, then

$$\left\| \sum_{\underline{\Pi}} f - \int_{a_1}^{b_1} f \right\| \leq \frac{\varepsilon}{2}$$

and

$$\left\| \sum_{\overline{\Pi}} f - \int_{a_2}^{b_2} f \right\| \leq \frac{\varepsilon}{2}.$$

If $A = -\infty$, let $\delta(-\infty) = \underline{\delta}(-\infty)$; if $B = +\infty$, let $\delta(+\infty) = \overline{\delta}(+\infty)$. Moreover, set

$$\delta(x) = \begin{cases} \min\left\{\underline{\delta}(x), \frac{1}{2}(c-x)\right\} & \text{if } x \in [A,c) \bigcap I\!R, \\ \min\left\{\overline{\delta}(x), \frac{1}{2}(x-c)\right\} & \text{if } x \in (c,B] \bigcap I\!R, \\ \min\{\underline{\delta}(c), \overline{\delta}(c)\} & \text{if } x = c, \end{cases}$$

and $P = \max\{\underline{P}, \overline{P}\}$. Take now any arbitrary bounded interval $[a,b] \subset [A,B]$, $[a,b] \supset [A,B] \bigcap [-P,P]$, and any δ-fine partition $\Pi = \{([u_k,v_k],\xi_k), k = 1,\ldots,p\}$ of $[a,b]$. Then necessarily $c \in (a,b)$. We now claim that there exists $k \in \{1,\ldots,p\}$ such that $c = \xi_k$, or $c = u_k$, or $c = v_k$. Otherwise there would be an interval $[u_j,v_j]$ such that $u_j < c < v_j$ and either $c < \xi_j < v_j$ or $u_j < \xi_j < c$. Since Π is δ-fine, we should get $[u_j,v_j] \subset (\xi_j - \delta(\xi_j), \xi_j + \delta(\xi_j))$ and thus $v_j - u_j < 2\,\delta(\xi_j)$. So $v_j - u_j < \xi_j - c$ if $\xi_j > c$ or $v_j - u_j < c - \xi_j$ if $\xi_j < c$. This would imply that ξ_j is outside (u_j,v_j), contradiction.

Thus we have:

$$\sum_{\Pi} f = \sum_{l=1}^{j-1} f(\xi_l)(v_l - u_l) + f(\xi_j)(v_j - u_j) + \sum_{l=j+1}^{p} f(\xi_l)(v_l - u_l) \tag{5}$$
$$= \sum_{l=1}^{j-1} f(\xi_l)(v_l - u_l) + f(\xi_j)(\xi_j - u_j) + f(\xi_j)(v_j - \xi_j) + \sum_{l=j+1}^{p} f(\xi_l)(v_l - u_l).$$

The quantity $S_a^c = \sum_{l=1}^{j-1} f(\xi_l)(v_l - u_l) + f(\xi_j)(\xi_j - u_j)$ is a Riemann sum for a suitable $\underline{\delta}$-fine partition of $[a, c]$, which is a bounded interval contained in $[A, c]$ and containing $[A, c] \bigcap [-\underline{P}, \underline{P}]$, by construction.
Analogously, the quantity $S_c^b = f(\xi_j)(v_j - \xi_j) + \sum_{l=j+1}^{p} f(\xi_l)(v_l - u_l)$ is a Riemann sum for a suitable $\overline{\delta}$-fine partition of $[c, b]$, which is a bounded interval contained in $[c, B]$ and containing $[c, B] \bigcap [-\overline{P}, \overline{P}]$. Thus we have:

$$\left\| S_a^c - \int_A^c f \right\| \le \frac{\varepsilon}{2}, \qquad \left\| S_c^b - \int_c^B f \right\| \le \frac{\varepsilon}{2},$$

and hence

$$\left\| \sum_{\Pi} f - \int_A^c f - \int_c^B f \right\| \le \varepsilon.$$

Thus the assertion follows. □

We now state two versions of the Cauchy criterion.

Theorem 2.7 *A map $f : [A, B] \to S$ is HK-integrable if and only if $\forall \varepsilon > 0 \ \exists$ a gauge γ such that for every γ-fine partition Π_1, Π_2 of $[A, B]$ we have*

$$\left\| \sum_{\Pi_1} f - \sum_{\Pi_2} f \right\| \le \varepsilon. \tag{6}$$

Proof: (see also [8]) The necessary part is straightforward.
We now turn to the sufficient part. By hypothesis, condition (6) holds even for $\varepsilon = \frac{1}{n}$, with $n \in I\!N$. Let γ_n be a corresponding gauge. Without loss of generality, we can suppose that

$$\gamma_{n+1}(x) \subset \gamma_n(x) \quad \forall\, x \in [A, B]. \tag{7}$$

Let $(\Pi_n)_n$ be a sequence of partitions of $[A,B]$ such that Π_n is γ_n-fine $\forall\, n \in I\!N$. From (7) it follows that, $\forall\, n,p \in I\!N$, every γ_{n+p}-fine partition is also γ_n-fine. Thus, in correspondence with $\varepsilon > 0$, let $\overline{n}$ be such that $\frac{1}{\overline{n}} \leq \varepsilon$: for $n \geq \overline{n}$ and $p \in I\!N$ we have:

$$\left\| \sum_{\Pi_{n+p}} f - \sum_{\Pi_n} f \right\| \leq \varepsilon.$$

Thus it follows that the sequence $\left(\sum_{\Pi_n} f \right)_n$ is Cauchy, and thus convergent, because of completeness of S. Let $I = \lim_n \sum_{\Pi_n} f$. Fix arbitrarily $\varepsilon > 0$. Then there exists an integer n^*, $n^* > \frac{2}{\varepsilon}$, such that

$$\left\| \sum_{\Pi_{n^*}} f - I \right\| \leq \frac{\varepsilon}{2}.$$

Let $\gamma = \gamma_{n^*}$. If Π is any γ-fine partition of $[A,B]$, then

$$\begin{aligned} \left\| \sum_{\Pi} f - I \right\| &\leq \left\| \sum_{\Pi} f - \sum_{\Pi_{n^*}} f \right\| + \left\| \sum_{\Pi_{n^*}} f - I \right\| \\ &\leq \frac{1}{n^*} + \frac{\varepsilon}{2} < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon. \end{aligned} \tag{8}$$

The assertion follows from (8). □

Theorem 2.8 *A map $f : [A,B] \to S$ is HK-integrable if and only if $\forall\, \varepsilon > 0 \;\exists$ a map $\delta \in \Delta$ and a positive real number P such that*

$$\left\| \sum_{\Pi_1} f - \sum_{\Pi_2} f \right\| \leq \varepsilon$$

whenever Π_1, Π_2 are δ-fine partitions of any bounded interval $[a,b]$, with $[a,b] \subset [A,B]$ and $[a,b] \supset [A,B] \bigcap [-P,P]$.

Proof: The proof is similar to the one of Theorem 2.7. □

We now prove a result about HK-integrability on subintervals.

Theorem 2.9 *Let $f : [A,B] \to S$ be HK-integrable, and $A < c < B$. Then $f_{|[A,c]}$ and $f_{|[c,B]}$ are HK-integrable too, and*

$$\int_A^B f = \int_A^c f + \int_c^B f. \tag{9}$$

Proof: By virtue of Theorem 2.7, $\forall\,\varepsilon > 0\ \exists$ a gauge γ on $[A, B]$ such that for all γ-fine partitions Π_1 and Π_2 of $[A, B]$ we have

$$\left\| \sum_{\Pi_1} f - \sum_{\Pi_2} f \right\| \le \varepsilon. \tag{10}$$

Set $\gamma_0 = \gamma_{|[A,c]}$ and let Π, $\Pi^{'}$ be any two γ_0-fine partitions of $[A, c]$. By virtue of the Cousin Lemma there exists a γ-fine partition Π_0 of $[c, B]$. Put $\Pi_1 = \Pi \bigcup \Pi_0$, $\Pi_2 = \Pi' \bigcup \Pi_0$. Then Π_1 and Π_2 are γ-fine partitions of $[A, B]$. Moreover, we get

$$\sum_{\Pi_1} f = \sum_{\Pi} f + \sum_{\Pi_0} f, \quad \sum_{\Pi_2} f = \sum_{\Pi'} f + \sum_{\Pi_0} f. \tag{11}$$

From (10) and (11) we have

$$\left\| \sum_{\Pi} f - \sum_{\Pi'} f \right\| \le \varepsilon. \tag{12}$$

From (12) and Theorem 2.7 it follows that $f_{|[A,c]}$ is HK-integrable. The proof of HK-integrability of $f_{|[c,B]}$ is analogous. The equality (9) follows from this and Proposition 2.6. □

We now prove the following:

Theorem 2.10 *Let $f : [A, B] \to S$ be an HK-integrable function. Let $A < c < B$. Then the function $g = f\,\chi_{[A,c]}$ is HK-integrable on $[A, B]$, and $\int_A^c f = \int_A^B g$.*

Proof: First of all, we note that $c \in I\!\!R$, and g is HK-integrable on $[A, c]$, because g coincides with f in $[A, c]$ and, by virtue of Theorem 2.9, f is HK-integrable on $[A, c]$. Moreover, it is easy to see that g is HK-integrable on $[c, B]$ and $\int_c^B g = 0$. So, by virtue of Proposition 2.6, we get that g is HK-integrable on $[A, B]$ and

$$\int_A^B g = \int_A^c g + \int_c^B g = \int_A^c f. \tag{13}$$

This concludes the proof. □.

Remark 2.11 In an analogous way it is possible to prove that $h = f\,\chi_{[c,B]}$ is HK-integrable on $[A, B]$ and $\int_c^B f = \int_A^B h$.

Corollary 2.12 *Let $f : [A, B] \to S$ be HK-integrable on $[A, B]$, and let $A < c < c' < B$. Then the map $l = f\,\chi_{[c,c']}$ is HK-integrable on $[A, B]$, and* $\int_c^{c'} f = \int_A^B l$.

Proof: First of all, we note that $c, c' \in I\!R$. Let $k = f_{|[A,c']}$: by virtue of Theorem 2.9, k is HK-integrable on $[A, c']$, and by Theorem 2.10, where the rôle of A, B, c is played by A, c', c respectively, the function

$$l' = k\,\chi_{[c,c']} = f_{|[A,c']}\,\chi_{[c,c']}$$

is HK-integrable on $[A, c']$, and $\int_c^{c'} f = \int_c^{c'} k = \int_A^{c'} l'$. Moreover, since l coincides with l' on $[A, c']$ and vanishes on $(c', B]$, then, thanks to Proposition 2.6, we get that l is HK-integrable on $[A, B]$ and $\int_A^B l = \int_A^{c'} l'$. From this the assertion follows. □

Now, given an interval $[a, b] \subset I\!R$, a partition $\Pi = \{([x_{k-1}, x_k], \xi_k), k = 1, 2, \ldots p\}$ and a point $c \in (a, b)$, if c coincides with some x_k, let $\Pi_1(\Pi_2)$ be the partition of all elements of Π which are contained in $[a, c]$ $([c, b])$ respectively, and put

$$\sum_{\Pi} {}_a^c\, f = \sum_{\Pi_1} f, \quad \sum_{\Pi} {}_c^b\, f = \sum_{\Pi_2} f.$$

If $c \in (x_{k-1}, x_k)$ for some $k = 1, \ldots, p$, then put

$$\sum_{\Pi} {}_a^c\, f = \sum_{l=1}^{k-1} f(\xi_l)(x_l - x_{l-1}) + f(c)(c - x_{k-1});$$

$$\sum_{\Pi} {}_c^b\, f = f(c)(x_k - c) + \sum_{l=k+1}^{p} f(\xi_l)(x_l - x_{l-1}).$$

In the sequel, when we will deal with the interval $[a, b]$ or $[A, B]$, sometimes we will write $\sum_\Pi {}_a^b\, f$, or $\sum_\Pi {}_A^B\, f$ respectively, instead of $\sum_\Pi f$, in order to avoid confusion. We now prove the following theorem (for the proof in the case $S = I\!R$, see [7], Lemma 2.8.1., pp. 56-57):

Theorem 2.13 *Let $[a, b] \subset I\!R$ be a bounded interval, $f : [a, b] \to S$ be a HK-integrable function, $\varepsilon > 0$, and $\delta : [a, b] \to I\!R^+$ such that, for every δ-fine partition Π' of $[a, b]$,*

$$\left\| \sum_{\Pi'} {}_a^b\, f - \int_a^b f \right\| \le \varepsilon. \tag{14}$$

Then δ is such that, $\forall\, c \in (a,b)$ and for every δ-fine partition Π of $[a,b]$,

$$\left\| \sum_{\Pi} {}_a^c\, f - \int_a^c f \right\| \le 2\varepsilon, \quad \left\| \sum_{\Pi} {}_c^b\, f - \int_c^b f \right\| \le 2\varepsilon. \tag{15}$$

Proof: Let Π be a δ-fine partition of $[a,b]$. By virtue of Theorem 2.9, f is HK-integrable in $[a,c]$, and thus there exists a function $\delta_c : [a,c] \to I\!R^+$ such that for every δ_c-fine partition Π'_c of $[a,c]$ we have:

$$\left\| \sum_{\Pi'_c} {}_a^c\, f - \int_a^c f \right\| \le \varepsilon. \tag{16}$$

Let now Π_c be a δ- and δ_c-fine partition of $[a,c]$. Moreover, let Π_0 be that partition of $[c,b]$ consisting of those elements $([x_{l-1},x_l],\xi_l)$ of Π such that the intervals $[x_{l-1},x_l]$ are contained in $[c,b]$ and eventually of (J,c), where J is the intersection of $[c,b]$ with that (eventual) interval $[x_{k-1},x_k]$ for which $x_{k-1} < c < x_k$. Let Π' be that partition consisting of the "union" of Π_c and Π_0: Π' is δ-fine, and we have:

$$\begin{aligned} & \sum_{\Pi} {}_c^b\, f - \int_c^b f = \sum_{\Pi_0} {}_c^b\, f - \int_c^b f \\ = \; & \sum_{\Pi'} {}_c^b\, f - \int_c^b f = \sum_{\Pi'} {}_a^b\, f - \int_a^b f \\ - \; & \left(\sum_{\Pi'} {}_a^c\, f - \int_a^c f \right) = \sum_{\Pi'} {}_a^b\, f - \int_a^b f \\ - \; & \left(\sum_{\Pi_c} {}_a^c\, f - \int_a^c f \right). \end{aligned}$$

By virtue of (14) and (16) we get:

$$\left\| \sum_{\Pi} {}_c^b\, f - \int_c^b f \right\| \le \left\| \sum_{\Pi'} {}_a^b\, f - \int_a^b f \right\| + \left\| \sum_{\Pi_c} {}_a^c\, f - \int_a^c f \right\| \le 2\varepsilon.$$

This proves the second inequality of (15). The proof of the first inequality of (15) is analogous. □

3 Comparison with the improper integral

We now prove that the HK-integral above defined contains the improper Riemann integral (For the real case, see [7], Theorem 2.9.3., pp. 61-63).

Theorem 3.1 *Let $a \in I\!R$, $f : [a, +\infty] \to S$ be HK-integrable on $[a, +\infty]$. Then f is HK-integrable on every interval $[a, b]$ with $a < b < +\infty$, and*

$$\lim_{b \to +\infty} \int_a^b f = \int_a^{+\infty} f.$$

Conversely, if $f : [a, +\infty] \to S$ is HK-integrable on every interval $[a, b]$ with $a < b < +\infty$ and there exists in S the limit $l = \lim_{b \to +\infty} \int_a^b f$, then f is HK-integrable on $[a, +\infty]$ and $\int_a^{+\infty} f = l$.

Proof: We begin with the first part of the theorem. Since $f : [a, +\infty] \to S$ is HK-integrable, then $\forall\, \varepsilon > 0\ \exists\, \delta : [a, +\infty] \to I\!R^+$ and $\exists\, P > |a|$, such that for each bounded interval $[d_1, d_2]$ with $[d_1, d_2] \subset [a, +\infty]$, $[d_1, d_2] \supset [a, +\infty] \bigcap [-P, P]$, and for every δ-fine partition Π of $[d_1, d_2]$ we have:

$$\left\| \sum_{\Pi} f - \int_a^{+\infty} f \right\| \leq \frac{\varepsilon}{2}. \tag{17}$$

Now, by virtue of Theorem 2.9, f is HK-integrable on $[a, b]$ for every $b \in (a, +\infty]$, and hence we get that $\forall\, \varepsilon > 0$, $\forall\, b \in (a, +\infty]$, $\exists\, \delta_1 : [a, b] \to I\!R^+$ such that for each δ_1-fine partition Π' of $[a, b]$ we get:

$$\left\| \sum_{\Pi'} f - \int_a^{b} f \right\| \leq \frac{\varepsilon}{2}. \tag{18}$$

Let us define $\delta_2 : [a, b] \to I\!R^+$ by setting $\delta_2(x) = \min\{\delta(x), \delta_1(x)\}$, and let Π be a δ_2-fine partition of $[a, b]$, $b > P$. Then, thanks to (17) and (18), $\forall\, \varepsilon > 0\ \exists\, P > 0$: $\forall\, b > P$,

$$\left\| \int_a^b f - \int_a^{+\infty} f \right\| \leq \left\| \sum_{\Pi} f - \int_a^{b} f \right\| + \left\| \sum_{\Pi} f - \int_a^{+\infty} f \right\| \leq \varepsilon.$$

Thus the first part is completely proved.

We now turn to the second part. By hypothesis, $\forall\, \varepsilon > 0$, $\exists\, P > 0$: $\forall\, b > P$ we get

$$\left\| \int_a^b f - l \right\| \leq \frac{\varepsilon}{2}. \tag{19}$$

Let now $(b_n)_n$ be a strictly increasing sequence of real numbers, such that $\lim_n b_n = +\infty$ and $b_1 = a$. We observe that, by virtue of Theorem 2.9, f is HK-integrable in $[b_n, b_{n+1}]$

for each n. So, $\forall\, \varepsilon > 0$ and $\forall\, n \in I\!N$, $\exists$ a function $\delta_n : [b_n, b_{n+1}] \to I\!R^+$ such that

$$\left\| \sum_{\Pi_n} f - \int_{b_n}^{b_{n+1}} f \right\| \leq \frac{\varepsilon}{2^{n+1}} \tag{20}$$

whenever Π_n is any δ-fine partition of $[b_n, b_{n+1}]$.

Let now $\delta : [a, +\infty] \to I\!R^+$ be such that, $\forall\, n \in I\!N$,

$$\begin{cases} \delta(\xi) \leq \delta_n(\xi) & \text{if } \xi \in [b_n, b_{n+1}], \\ \\ [\xi - \delta(\xi), \xi + \delta(\xi)] \subset (b_n, b_{n+1}) & \text{if } \xi \in (b_n, b_{n+1}), \\ \\ (b_n - \delta(b_n), b_n + \delta(b_n)) \subset (b_{n-1}, b_{n+1}). \end{cases} \tag{21}$$

Choose now arbitrarily $b > P$. If $b_N < b \leq b_{N+1}$ and $\Pi = \{([x_{k-1}, x_k], \xi_k), k = 1, 2, \dots p\}$ is a partition of $[a, b]$, then each b_n, with $n \leq N$, must belong to some interval $[x_{k-1}, x_k]$. So, either b_n coincides with some x_k's, or $b_n \in (x_{k-1}, x_k)$. In this last case, from (21) and the fact that Π is δ-fine it follows that $\xi_k \notin (b_n, b_{n+1})$, otherwise

$$[x_{k-1}, x_k] \subset (\xi_k - \delta(\xi_k), \xi_k + \delta(\xi_k)) \subset (b_n, b_{n+1}) :$$

this is a contradiction. Analogously, $\xi_k \notin (b_{n-1}, b_n)$, and in general, if $j \in I\!N$ is such that $b_j \in (x_{k-1}, x_k)$, we have necessarily $\xi_k \notin (b_{j-1}, b_j)$, $\xi_k \notin (b_j, b_{j+1})$: otherwise $[x_{k-1}, x_k] \subset (b_{j-1}, b_j)$ or $[x_{k-1}, x_k] \subset (b_j, b_{j+1})$: this is absurd. Thus ξ_k does coincide with some b_{j_0}. From the third condition in (21) and the fact that Π is δ-fine it follows that

$$\begin{aligned} [x_{k-1}, x_k] &\subset (\xi_k - \delta(\xi_k), \xi_k + \delta(\xi_k)) \\ &= (b_{j_0} - \delta(b_{j_0}), b_{j_0} + \delta(b_{j_0})) \subset (b_{j_0-1}, b_{j_0+1}). \end{aligned} \tag{22}$$

But we know that, by hypothesis, $b_n \in (x_{k-1}, x_k)$, and from (22) it follows that $j_0 = n$ and that no b_j but b_n belongs to (x_{k-1}, x_k). So, all the b_n's do coincide either with some x_k or with some ξ_k. So, Π is the partition of $[a, b]$ "determined" by the x_k's and the b_n's. We have:

$$\sum_{\Pi} {}_a^b\, f = \sum_{n=1}^{N-1} \left(\sum_{\Pi} {}_{b_n}^{b_{n+1}}\, f \right) + \sum_{\Pi} {}_{b_N}^{b}\, f. \tag{23}$$

Since the restriction of Π to $[b_n, b_{n+1}]$ is δ_n-fine, from (20) it follows that

$$\sum_{n=1}^{N-1} \left\| \sum_{\Pi} {}_{b_n}^{b_{n+1}} f - \int_{b_n}^{b_{n+1}} f \right\| \leq \frac{\varepsilon}{2}. \tag{24}$$

From (19), (23) and (24) we have:

$$\left\| \sum_{\Pi} {}_a^b \, f - l \right\| \leq \frac{\varepsilon}{2} + \frac{\varepsilon}{2} + \left\| \sum_{\Pi} {}_{b_N}^{b} f - \int_{b_N}^{b} f \right\|.$$

Since the restriction of Π to $[b_N, b]$ is δ_N-fine, then Π can be "extended" to a δ_N-fine partition Π' of $[b_N, b_{N+1}]$. By Theorem 2.13, where the rôles of $[a, b]$ and c are played by $[b_N, b_{N+1}]$ and b respectively, we get

$$\left\| \sum_{\Pi} {}_{b_N}^{b} f - \int_{b_N}^{b} f \right\| \leq \frac{\varepsilon}{2^N} < \varepsilon.$$

From this the assertion follows. $\square$

Remark 3.2 We observe that theorems similar to Theorem 3.1 hold even if we consider open, semi-open and/or left halflines, $I\!R$ or $\widetilde{I\!R}$, instead of $[a, +\infty]$.

We now prove that every simple measurable function defined on $I\!R$, and assuming values different from zero only on a set of finite Lebesgue measure, is HK-integrable according to our definition, and in this case our integral coincides with the usual one. To do this, thanks to Proposition 2.5, it is sufficient to prove the following:

Theorem 3.3 *Let $E \subset I\!R$ be a Lebesgue measurable set with $|E| < +\infty$, $r \in S$, and χ_E be the characteristic function associated with E. Then the function $\chi_E\, r$ is HK-integrable, and $\int_{-\infty}^{+\infty} \chi_E\, r = |E| r$.*

Proof: By virtue of [7], p. 136, we know that the theorem is true in the particular case $S = I\!R$ and $r = 1$. Thus for every $\varepsilon > 0$ there exists a gauge γ, defined on $I\!R$, such that for each γ-fine partition Π of $I\!R$ we get

$$\left| \sum_{\Pi} \chi_E - |E| \right| \leq \varepsilon. \tag{25}$$

Moreover, it is easy to see that for each partition Π of $I\!R$ we have

$$\sum_{\Pi} \chi_E\, r = \left(\sum_{\Pi} \chi_E \right) r. \tag{26}$$

The assertion follows from (25), (26) and (uniform) continuity of the "norm" map in Banach spaces. $\square$

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