Comparison between different types of abstract integrals in Riesz spaces

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Abstract

A comparison among different types of integral in Riesz spaces is given.
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1 Introduction.

In a previous paper (see [4]), we introduced a "monotone-type" integral for extended-real valued maps, with respect to Riesz-space-valued finitely additive function (see also [9]), [12]). More precisely, given a mean $\mu : \mathcal{A} \to R$ and a measurable function $f : X \to \widetilde{I\!R}_0^+$, we say that f is integrable in the monotone sense, or (M)-integrable, if there exists in R the

$$(o) - \lim_{a \to +\infty} \int_0^a u(t)\ dt,$$

where $u(t) \equiv \mu(\{x \in X : f(x) > t\})\ dt,\ \forall\ t \in I\!R^+$, and the integral is intended as a Riemann-type integral.
In this paper, firstly we show that the Riemann integral is equivalent to the Mengoli-Cauchy integral, and after we compare the monotone integral with other types of integrals.
In particular, we introduce a Dunford-Schwartz-type integral (see also [11]), similar to the one introduced in [3], but with some differences, and we prove that it coincides with the monotone integral, by virtue of the Vitali-type theorem for the (M)-integral given in [4].

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Furthermore, some comparisons with pointwise-type integral and Chojnacki-integral are investigated.

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2 Preliminaries.

Definition 2.1 Let X be any set, R a Dedekind complete Riesz space, $\mathcal{A} \subset \mathcal{P}(X)$ an algebra. A map $\mu : \mathcal{A} \to R$ is said to be a *mean* if $\mu(A) \geq 0$, $\forall\ A \in \mathcal{A}$, and $\mu(A \cup B) = \mu(A) + \mu(B)$, whenever $A \cap B = \emptyset$. A mean μ is *countably additive* (or *σ-additive*) if $\mu(\cap_n A_n) = \inf_n \mu(A_n)$, whenever $(A_n)_n$ is a decreasing sequence in $\mathcal{A}$, such that $\cap_n A_n \in \mathcal{A}$.

Definition 2.2 A net $\{x_\alpha\}_{\alpha\in\Lambda}$ is said to be (o)-*convergent* (or simply *convergent*) if there exist in R the quantities

$$(o) - \limsup_{\alpha\in\Lambda} x_\alpha \equiv \inf_\alpha \sup_{\lambda\geq\alpha} x_\lambda$$

and

$$(o) - \liminf_{\alpha\in\Lambda} x_\alpha \equiv \sup_\alpha \inf_{\lambda\geq\alpha} x_\lambda,$$

and they coincide;
convergent to x if $x = (o) - \limsup_{\alpha\in\Lambda} x_\alpha = (o) - \liminf_{\alpha\in\Lambda} x_\alpha$; in this case, we write $(o) - \lim_{\alpha\in\Lambda} x_\alpha = x$, and say that x is the (o)-limit of $\{x_\alpha\}$.

Definition 2.3 A net $\{x_\alpha\}_\alpha$ is said to be (o)-*Cauchy* (or simply *Cauchy*) if

$$\limsup_{\alpha,\ \beta} |x_\alpha - x_\beta| = 0.$$

Definition 2.4 A Riesz space R is called $[\sigma]$-*Dedekind complete* if every [countable] subset of R, bounded from above, has supremum in R.

The following result justifies the above definition:

Proposition 2.5 *Let R be a Dedekind complete Riesz space. Then, a net in R is convergent if and only if it is Cauchy* (see also [15]).

3 An equivalent definition of Riemann-integral for Riesz-space-valued functions.

In [4] we defined the integral $\int_0^a u(t)\ dt$ as a Riemann - type integral. This integral can be defined also as a "Mengoli-Cauchy" type integral. We will show that the "Riemann"-integral and the "Mengoli-Cauchy"-integral coincide.

Definition 3.1 Given an interval $[a,b] \subset I\!R$, we call *division of* $[a,b]$ a finite set $\{x_0,\ x_1,\ldots,\ x_n\} \subset [a,b]$, where $x_0 = a,\ x_n = b$, and $x_i < x_{i+1},\ \forall\ i = 0,\ldots,n$. We call *mesh* of D the quantity $(\delta(D)) \equiv \max_i\ (x_{i+1} - x_i)$. We say that $D_1 \geq D_2$ if $\delta(D_1) \leq \delta(D_2)$.

We now recall the definition of "Riemann-integral" given in [4].

Definition 3.2 Let R be a Dedekind complete Riesz space, and $u : [a,b] \to R$ a bounded map. We call *upper integral* [resp. *lower integral*] of u the element of R given by

$$\inf_{v \in V_u} \int_a^b\ v(t)\ dt\ \ [\sup_{s \in S_u}\ \int_a^b\ s(t)\ dt],$$

where

$$\begin{aligned} V_u &\equiv \{v : v \text{ is a step function },\ v(t) \geq u(t),\ \forall\ t \in [a,b]\} \\ S_u &\equiv \{s : s \text{ is a step function },\ s(t) \leq u(t),\ \forall\ t \in [a,b]\}. \end{aligned}$$

We say that a bounded function $u : [a,b] \to R$ is *Riemann-integrable* (or (R)-*integrable*), if its lower integral coincides with its upper integral, and, in this case, we call *integral of* u (and write $\int_a^b\ u(t)\ dt$) the common value of them. We also indicate it by

$$(R) - \int_a^b\ u(t)\ dt.$$

Definition 3.3 Let $[a,b] \subset I\!R, R$ be as above, and $u : [a,b] \to R$ be a map. We say that u is *Mengoli-Cauchy integrable* ((MC)-*integrable*) if there exists an element $I \in R$ and a sequence $(p_n)_n,\ p_n \downarrow 0$, such that,

$$\sup_{\delta(D) \leq \frac{1}{n}}\ |\sum_{i=1}^{k}\ u(z_i)(x_i - x_{i-1}) - I| \leq p_n,\ \forall\ z_i \in [x_{i-1}, x_i]\ (i = 1,\ldots,k),$$

and we write $(MC) - \int_a^b\ u(t)\ dt \equiv I$.

The following result holds:

Theorem 3.4 *With the same notations as above, let* $u : [a,b] \to R$ *be Mengoli- Cauchy integrable. Then* u *is bounded.*

The proof is straightforward.

Theorem 3.5 *Let* $u : [a,b] \to R$ *be Mengoli-Cauchy integrable. Then,* u *is Riemann integrable, and*

$$(R) - \int_a^b\ u(t)\ dt = (MC) - \int_a^b\ u(t)\ dt.$$

Proof. Let $I,\ p_n \downarrow 0$ as in Definition 3.3. Let $D \equiv \{x_0, x_1, \ldots, x_k\}$ be such that $\delta(D) < \frac{1}{n}$. We consider the following two functions associated with D :

$$s_0(x) = \begin{cases} \inf_{t\in]x_{i-1},x_i[}\ u(t), & x \in]x_{i-1}, x_i[,\ i = 1, \ldots, k \\ u(x_i), & x = x_i,\ i = 0, 1, \cdots k; \end{cases}$$

$$v_0(x) = \begin{cases} \sup_{t\in]x_{i-1},x_i[}\ u(t), & x \in]x_{i-1}, x_i[,\ i = 1, \ldots, k \\ u(x_i), & x = x_i,\ i = 0, 1, \cdots k; \end{cases}$$

By hypothesis, we have:

$$I - p_n \leq \sum_{i=1}^{k}\ u(z_i)\ (x_i - x_{i-1}) \leq I + p_n.$$

Then, keeping fixed z_i for $i \geq 2$, and taking the suprema as z_1 varies, we get

$$I - p_n \leq \sum_{i\geq 2}\ u(z_i)\ (x_i - x_{i-1}) + v_0(\frac{x_1 + x_0}{2})\ (x_1 - x_0) \leq I + p_n.$$

Now, we repeat the same procedure, keeping fixed z_i for $i \geq 3$, and so on, until we obtain

$$I - p_n \leq \int_a^b\ v_0(t)\ dt \leq I + p_n.$$

Similarly we can get

$$I - p_n \leq \int_a^b\ s_0(t)\ dt \leq I + p_n$$

and hence

$$\int_a^b\ v_0(t)\ dt - I \leq p_n,$$

$$I - \int_a^b\ s_0(t)\ dt \leq p_n,$$

from which we obtain

$$|^* \int_a^b\ u(t)\ dt - I| \leq p_n,$$

$$|_* \int_a^b\ u(t)\ dt - I| \leq p_n,$$

$$^* \int_a^b\ u(t)\ dt -_* \int_a^b\ u(t)\ dt \leq 2\ p_n.$$

By arbitrariness of D, we find that

$$^*\int_a^b u(t)\ dt =_* \int_a^b u(t)\ dt = I \square.$$

Theorem 3.6 *Let* $u : [a,b] \to R$ *be Riemann-integrable. Then,* u *is Mengoli-Cauchy integrable, and*

$$(MC) - \int_a^b u(t)\ dt = (R) - \int_a^b u(t)\ dt.$$

Proof: Fix arbitrarily $s \in S_u$ and $v \in V_{u-s}$. Choose a division $D^* \equiv \{c_0, c_1, \ldots, c_{N-1}\}$, such that both s and v are constant in $]c_{j-1}, c_j[,\ \forall\ j$, and put $M \equiv \sup_{x\in[a,b]}\ u(x)$. Fix $n \in I\!N$, and consider a division $D \equiv \{x_0, x_1, \ldots, x_k\}$, such that $\delta(D) \le \frac{1}{n}$. Let z_i be in $[x_{i-1}, x_i]$.
Now define the step function $\theta : [a,b] \to R$ by setting

$$\theta(x) = \begin{cases} u(z_i), \text{ if } x \in [x_{i-1}, x_i[,\ i = 1, 2, \ldots, k \\ \\ u(b), \text{ if } x = b. \end{cases}$$

If $x \in [x_{i-1}, x_i] \subset]c_{j-1}, c_j[$ for some suitable j, then we have:

$$\begin{aligned} |u(x) - \theta(x)| &= |u(x) - u(z_i)| \le [u(x) - s(x)] + |s(x) - u(z_i)| \le \\ &\le v(x) + \sup_{x\in[x_{i-1},x_i]} [u(x) - s(x)] \le 2\ v(x). \end{aligned}$$

If $x \in [x_{i-1}, x_i] \not\subset]c_{j-1}, c_j[\ \forall\ j$, then

$$|u(x) - \theta(x)| \le |u(x)| + |\theta(x)| \le 2\ M.$$

So,

$$\begin{aligned} &|(R) - \int_a^b u(x)\ dx - \sum_{i=1}^n u(z_i)(x_i - x_{i-1})| = |(R) - \int_a^b u(x)\ dx - \int_a^b \theta(x)\ dx| \le \\ \le &\int_a^b |u(x)\ - \theta(x)|\ dx \le 2 \int_a^b v(x)\ dx + 2\ N\ \frac{1}{n}\ M. \end{aligned}$$

Thus, we get:

$$\begin{aligned} 0 &\le \sup_{\delta(D)\le\frac{1}{n}} |(R) - \int_a^b u(x)\ dx - \sum_{i=1}^n u(z_i)(x_i - x_{i-1})| \le \\ &\le 2 \int_a^b v(x)\ dx + 2\ N\ \frac{1}{n}\ M. \end{aligned}$$

By arbitrariness of v and (R)-integrability of $u - s$, we obtain:

$$\begin{aligned} 0 &\leq (o) - \limsup_{n\to+\infty} \sup_{\delta(D)\leq\frac{1}{n}} |(R) - \int_a^b u(x)\ dx - \sum_{i=1}^n u(z_i)(x_i - x_{i-1})| \leq \\ &\leq 2 \inf_{v\in V_{u-s}} \int_a^b v(x)\ dx + (o) - \lim_{n\to+\infty} 2\ N\ \frac{1}{n}\ M = \\ &= 2 \int_a^b [u(x) - s(x)]\ dx = 2 \int_a^b u(x)\ dx - 2 \int_a^b s(x)\ dx. \end{aligned}$$

By arbitrariness of s and (R)-integrability of u, we get:

$$\begin{aligned} 0 &\leq (o) - \limsup_{n\to+\infty} \sup_{\delta(D)\leq\frac{1}{n}} |(R) - \int_a^b u(x)\ dx - \sum_{i=1}^n u(z_i)(x_i - x_{i-1})| \leq \\ &\leq 2 \int_a^b u(x)\ dx - 2 \sup_{s\in S_u} \int_a^b s(x)\ dx = 0. \end{aligned}$$

So,

$$(o) - \lim_{n\to+\infty} \sup_{\delta(D)\leq\frac{1}{n}} |(R) - \int_a^b u(x)\ dx - \sum_{i=1}^n u(z_i)(x_i - x_{i-1})| = 0$$

uniformly with respect to the z_i. □

4 Bochner-type integrals of real-valued function.

With the same notations as in the previous section, if $R_1 = I\!R$, and $R \equiv R_2 = R_3$ is a Dedekind complete Riesz space, we can formulate the definition of convergence in measure and develop our theory in a way, which is somewhat different from the one in [3].
In [3], definitions of convergence in measure, integral, and so on were introduced; here, we give other definitions of "convergence in measure", "integral", etc. and compare them with the former.

Definition 4.1 Let X be any set, $\mu : \mathcal{A} \to R$ a positive finitely additive set function. We say that a sequence $(f_n)_n$ of extended real-valued functions, defined on X, (o)-*converges in measure to* f *if*

$$(o) - \lim_n \mu(\{x \in X : |f_n(x) - f(x)| > \varepsilon\}) = 0,\ \forall\ \varepsilon > 0;$$

(B)-converges *in measure* to f if there exist two sequences $(p_n)_n$, $(q_n)_n$, $R \ni p_n \downarrow 0$, $I\!R \ni q_n \downarrow 0$, such that

$$\mu(\{x \in X : |f_n(x) - f(x)| > q_n\}) \leq p_n,\ \forall\ n \in I\!N$$

(see [3]).

Definition 4.2 Let $(f_n : X \to \widetilde{\mathbb{R}})_n$ be a sequence of simple functions. We say that $(f_n)_n$ is *equiintegrable* if

$$\sup_n \int_X |f_n| \, d\mu \in R, \tag{1}$$

and

$$(o) - \lim_n \sup_{k \geq n} \left(\int_{A_n} |f_k| \, d\mu \right) = 0, \tag{2}$$

whenever $(o) - \lim_k \mu(A_k) = 0$.

Now, we compare (B)-convergence in measure with (o)-convergence in measure. We begin with the following:

Definition 4.3 Let R be any Riesz space, and let $u \in R$, $u \geq 0$. We say that u has the *Egoroff property* if, for each double sequence $(u_{n,k})_{n,k}$ in R, satisfying $u \geq u_{n,k} \downarrow 0$ $(k \to +\infty,\ n = 1, 2, \ldots)$, there exists a sequence $(v_n)_n$ in R, $v_n \downarrow 0$, with the property that, for all $n \in \mathbb{N}$, there exists $k = k_n \in \mathbb{N}$, such that $u_{n,k_n} \leq v_n$.
We say that a Riesz space R has the *Egoroff property* (or is *Egoroff*) if every positive element of R has the Egoroff property.

We note that, if Σ is any finite or countable set, then $\mathbb{R}^\Sigma$ is Egoroff, but, if the cardinality of Σ is greater or equal to c, then $\mathbb{R}^\Sigma$ is not Egoroff. Moreover, if $R = L^p(\lambda)$, where $0 \leq p \leq \infty$, and λ is a countably additive σ-finite real-valued measure, then R is Egoroff. Furthermore, every solid subspace of an Egoroff space R is Egoroff too (see also [14]). The following result gives the comparison announced:

Theorem 4.4 *Let $\mu : \mathcal{A} \to R$ be a positive finitely additive set function. If $(f_n : X \to \mathbb{R})_n$ (B)-converges in measure to $f \in \mathbb{R}^X$, then $(f_n)_n$ (o)-converges in measure to f.*
Moreover, if $\mu(X)$ has the Egoroff property, and $(f_n)_n$ (o)-converges in measure to f, then $(f_n)_n$ (B)-converges in measure to f.

Proof: We begin with proving the first part of the assertion. Fix $\varepsilon > 0$, and let $(p_n)_n$ and $(q_n)_n$ satisfy the definition of (o)-convergence in measure. Then, there exists a natural number $\overline{n}(\varepsilon)$ such that $q_n < \varepsilon,\ \forall\, n \geq \overline{n}$, and so

$$\mu(\{x \in X : |f_n(x) - f(x)| > \varepsilon\}) \leq \mu(\{x \in X : |f_n(x) - f(x)| > q_n\}).$$

Define $r_n = r_n(\varepsilon) \equiv \begin{cases} \mu(X) & \text{if } n < \overline{n}(\varepsilon) \\ p_n \wedge \mu(X), & \text{if } n \geq \overline{n}(\varepsilon). \end{cases}$

Thus, for every $n \in I\!N$, we have:

$$\mu(\{x \in X : |f_n(x) - f(x)| > \varepsilon\}) \le r_n \downarrow 0.$$

Now, we prove the second part. By hypothesis, there exists a double sequence $(r_{n,k})_{n,k},\ R \ni r_{n,k} \downarrow 0\ (k \to +\infty,\ n = 1, 2, \ldots)$ such that

$$\mu(\{x \in X : |f_k(x) - f(x)| > \frac{1}{n}\}) \le r_{n,k},\ \forall\ n, k \in I\!N.$$

Put $u_{n,k} \equiv r_{n,k} \wedge \mu(X),\ \forall\ n, k$. Of course, $u_{n,k} \le \mu(X)$ for every n, k, and $u_{n,k} \downarrow 0\ (k \to +\infty,\ n = 1, 2, \ldots)$. As $\mu(X)$ has the Egoroff property, then there exists a sequence $(v_n)_n,\ v_n \downarrow 0$, such that, $\forall\ n,\ \exists\ k = k(n) \in I\!N : u_{n,k(n)} \le v_n$.

For $k \in I\!N$, set $p_k \equiv \begin{cases} \mu(X) \vee v_1, & \text{if } 1 \le k \le k_1 \\ v_n, & \text{if } k_n \le k \le k_{n+1},\quad n \in I\!N. \end{cases}$

Moreover, put $q_k \equiv \begin{cases} 1, & \text{if } 1 \le k \le k_1 \\ \frac{1}{n}, & \text{if } k_n \le k \le k_{n+1},\quad n \in I\!N. \end{cases}$

It is easy to check that

$$\mu(\{x \in X : |f_k(x) - f(x)| > q_k\}) \le p_k,\ \forall\ k \in I\!N,$$

and $p_k,\ q_k \downarrow 0$. So, the theorem is completely proved.

Definition 4.5 A sequence $(f_n)_n$ of measurable functions is said to be *Cauchy in measure* if

$$(o) - \lim_n\ \mu(\{x \in X : |f_n(x) - f_{n+p}(x)| > \varepsilon\}) = 0$$

uniformly with respect to $p \in I\!N,\ \ \forall\ \varepsilon > 0$.

Definition 4.6 A sequence (f_n) of simple functions *converges in* L^1 to the simple function f if

$$(o) - \lim_n \int_X |f_n - f|\ d\mu = 0.$$

Convergence in L^1 can be characterized as follows:

Proposition 4.7 *Let f_n and f be as above. Then, $(f_n)_n$ converges in L^1 to f if and only if*

$$(o) - \lim_n \int_A f_n \, d\mu = \int_A f \, d\mu$$

uniformly with respect to $A \in \mathcal{A}$.

Proof: The "only if" part is easy.
We now turn to the "if" part. By hypothesis, there exists a sequence $(p_n)_n, \; R \ni p_n \downarrow 0$, such that

$$|\int_A f_n \, d\mu - \int_A f \, d\mu| \leq p_n, \; \forall \, n \in I\!N, \; \forall \, A \in \mathcal{A}.$$

For each $n \in I\!N$, let $A_n \equiv \{x \in X : f_n(x) \geq f(x)\}$. We have:

$$\int_X |f_n - f| \, d\mu = \int_{A_n} (f_n - f) \, d\mu + \int_{A_n{}^c} (f - f_n) \, d\mu =$$

$$= |\int_{A_n} f_n \, d\mu - \int_{A_n} f \, d\mu| + |\int_{A_n{}^c} f \, d\mu - \int_{A_n{}^c} f_n \, d\mu| \leq 2 \, p_n,$$

that is the assertion. □

Definition 4.8 A sequence (f_n) of simple functions is *Cauchy in L^1* if

$$(o) - \lim_n \int_X |f_n - f_{n+p}| \, d\mu = 0$$

uniformly with respect to $p \in I\!N$.

Analogously as in Proposition 4.7, one can prove the following:

Proposition 4.9 *Let $(f_n)_n$ be as above. Then, $(f_n)_n$ is Cauchy in L^1 if and only if the sequence $(\int_A f_n \, d\mu)_n$ is Cauchy uniformly with respect to $A \in \mathcal{A}$.*

Definition 4.10 Under the same notations as above, a map f is said to be *integrable* if there exists a sequence $(f_n)_n$ of simple functions, convergent in measure to f and Cauchy in L^1. In this case, we define

$$\int_A f \, d\mu \equiv (o) - \lim_n \int_A f_n \, d\mu, \; \forall \; A \in \mathcal{A}.$$

Definition 4.11 If f is integrable, put

$$\int_A f \, d\mu \equiv \; (o) - \lim_{n \to \infty} \int_A f_n \, d\mu, \; \forall \; A \in \mathcal{A},$$

where $(f_n)_n$ is a sequence of simple function, convergent in measure to f and Cauchy in L^1.

Now, we prove that the integral in 4.11 is well-defined.

Theorem 4.12 *Let f be an integrable function, and $(f_n)_n$ as in* 4.11. Then the limit $(o)-\lim_{n\to\infty}\int_A f_n\, d\mu$ exists uniformly with respect to $A \in \mathcal{A}$ and does not depend on the choice of $(f_n)_n$.

Proof (see also [11])**:** Let $(f_n^1)_n$, $(f_n^2)_n$ be two sequences of simple maps, convergent in measure to the same limit f and Cauchy in L^1. Then, there exists $(q_n^i)_n,\ R \ni q_n^i \downarrow 0$, such that

$$|\int_A f_n^i\, d\mu - \int_A f_m^i\, d\mu| \le \int_X |f_n^i - f_m^i|\, d\mu \le q_n^i \le q_n^1 + q_n^2\ (i=1,2),$$

$\forall\, n \in I\!N,\ \forall\, m \ge n,\ \forall\, A \in \mathcal{A}$.
As R is Dedekind complete, then the sequences $(\int_A f_n^i\, d\mu)_n$ $(i = 1,2)$ are (o)-convergent, uniformly with respect to $A \in \mathcal{A}$. We denote by $l_i\ (A)$ their $(o)-$limits. For every $A \in \mathcal{A}$, let $P_n(A) \equiv \int_A p_n\, d\mu$, where $p_n(x) \equiv |f_n^1(x) - f_n^2(x)|,\ \forall\, x \in X$. The sequence $(p_n)_n$ converges in measure to 0, and it is easy to see that $(P_n(A))_n$ is Cauchy uniformly with respect to A; then, $(o)-\lim_n P_n(A)$ exists in R, uniformly with respect to $A \in \mathcal{A}$: we denote this limit by $P(A)$. As the integral of simple functions is absolutely continuous, we have that

$$[(o)-\lim_k \mu(E_k) = 0] \Longrightarrow [(o)-\lim_k P_n(E_k) = 0,\ \forall\, n \in I\!N.]$$

Now, we prove that $(o)-\lim_k P(E_k) = 0$. Fix arbitrarily $n, k \in I\!N$. Then, there exist some sequences in R, $(t_n)_n,\ (r_{n,k})_{n,k}$, such that $t_n \downarrow 0,\ r_{n,k} \downarrow_k 0$ for all fixed $n \in I\!N$, and

$$|P(E_k) - P_n(E_k)| \le t_n,\ P_n(E_k) \le r_{n,k},\ \forall\, n, k.$$

Thus, $\forall\, n \in I\!N$, we have:

$$0 \le (o)-\limsup_k\ (P(E_k)) \le (o)-\limsup_k\ |P(E_k) - P_n(E_k)| +$$

$$+(o)-\limsup_k\ (P_n(E_k)) \le t_n + \inf_k\ r_{n,k} = t_n.$$

By arbitrariness of n, we get $(o)-\lim_k\ (P(E_k)) = 0$.
By convergence in measure of $(p_n)_n$ to 0, for every fixed $\varepsilon > 0$ and $n \in I\!N$, we have:

$$\begin{aligned}0 \ &\le\ P(X) = P(\{x \in X : p_n(x) > \varepsilon\}) + [P(\{x \in X : p_n(x) \le \varepsilon\}) - P_n(\{x \in X : p_n(x) \le \varepsilon\})] + \\ &+\ P_n(\{x \in X : p_n(x) \le \varepsilon\}) \le v_n + w_n + \varepsilon\ \mu(X),\end{aligned}$$

for some suitable sequences $(v_n)_n,\ (w_n)_n$ in R, such that $v_n \downarrow 0 \downarrow w_n$. Taking the infima with respect to n, and by arbitrariness of ε, we obtain $P(X) = 0$. As $0 \leq P_n(A) \leq P_n(X)\ \forall\ n \in I\!N,\ \forall\ A \in \mathcal{A}$, we get: $P(A) = 0,\ \forall\ A \in \mathcal{A}$. So, $\forall\ n \in I\!N,\ \forall A \in \mathcal{A}$, we get:

$$\sup_A\ |l_1(A) - l_2(A)| \leq |\int_A f_n^1\ d\mu - l_1(A)| + |l_2(A) - \int_A f_n^2\ d\mu| +$$

$$+|\int_A f_n^1\ d\mu - \int_A f_n^2\ d\mu| \leq a_n + b_n + \int_A p_n\ d\mu \leq a_n + b_n + c_n,$$

for some suitable sequences $(a_n)_n,\ (b_n)_n,\ (c_n)_n,\ a_n \downarrow 0,\ b_n \downarrow 0,\ c_n \downarrow 0$. Taking the infima, we get:

$$\sup_A |l_1(A) - l_2(A)| \leq \inf_n\ (a_n + b_n + c_n) = 0.$$

Thus, $l_1(A) = l_2(A),\ \forall\ A \in \mathcal{A}$. □

Remark 4.13 It is readily seen that the integral introduced in 4.11 is a linear monotone functional and a finitely additive set function.

Lemma 4.14 *Under the same notations as above, let f be an integrable function, and $(f_n)_n$ a sequence of simple function, convergent in measure to f and Cauchy in L^1. Then,*

$$(o) - \lim_n\ \int_X\ |f_n - f|\ d\mu = 0.$$

Proof: As $(f_n)_n$ is Cauchy in L^1, there exists a sequence $(y_n)_n,\ R \ni y_n \downarrow 0$, such that

$$\int_X\ |f_n - f_m|\ d\mu \leq y_n.$$

Fix $n \in I\!N$. As $(f_m)_m$ converges in measure to f, then $(|f_n - f_m|)_m$ converges in measure to $|f_n - f|$. Moreover, it is easy to check that $(|f_n - f_m|)_m$ is Cauchy in L^1. So,

$$\int_A\ |f_n - f|\ d\mu = (o) - \lim_m\ \int_A\ (|f_n - f_m|)\ d\mu,$$

uniformly with respect to $A \in \mathcal{A}$, and thus

$$\int_X\ |f_n - f|\ d\mu \leq y_n,$$

that is the assertion. □

Lemma 4.15 *Let f be an integrable function, and let $(A_{n,\lambda})_{n \in I\!N, \lambda \in \Lambda}$ be a family of subsets of X, such that*

$$(o) - \lim_n\ \left(\sup_\lambda\ \mu(A_{n,\lambda})\right) = 0.$$

Then,

$$(o) - \lim_n \left(\sup_\lambda \int_{A_{n,\lambda}} |f| \; d\mu \right) = 0.$$

Proof: Let $(f_h)_{h\in I\!N}$ be as in Lemma 4.14. There exist some sequences $(z_n)_n,\ (d_h)_h,\ R \ni z_n \downarrow 0,\ R \ni d_h \downarrow 0$, and there exists some real numbers $v_h,\ h \in I\!N$, such that, for all n, k, λ, we have:

$$\int_{A_{n,\lambda}} |f| \; d\mu \le \int_X |f - f_h| \; d\mu + \int_{A_{n,\lambda}} |f_h| \; d\mu \le d_h + v_h \; z_n,$$

and hence

$$\sup_{\lambda\in\Lambda} \int_{A_{n,\lambda}} |f| \; d\mu \le d_h + v_h \; z_n;$$

thus

$$0 \le (o) - \limsup_n \left(\sup_{\lambda\in\Lambda} \int_{A_{n,\lambda}} |f| \; d\mu \right) \le d_h + (o) - \limsup_n \; v_h \; z_n, \; \forall \; h$$

and therefore

$$0 \le (o) - \limsup_n \left(\sup_{\lambda\in\Lambda} \int_{A_{n,\lambda}} |f| \; d\mu \right) \le \inf_h \; d_h = 0,$$

that is

$$(o) - \lim_n \left(\sup_{\lambda\in\Lambda} \int_{A_{n,\lambda}} |f| \; d\mu \right) = 0.\Box$$

We now state the following:

Theorem 4.16 *Let $(f_n)_n$ be a sequence of simple functions of R^X, convergent in measure to $f \in R^X$. Then, the following are equivalent:*

4.16.1.) *$(f_n)_n$ is Cauchy in L^1*

4.16.2.) *f is integrable and $(f_n)_n$ converges in L^1 to f.*

4.16.3.) $\sup_n \int_X f_n \; d\mu \in R$; *and*

$$(o) - \lim_n \left[\sup_\lambda \left(\sup_{m \ge n} \int_{A_{n,\lambda}} |f_m| \; d\mu \right) \right] = 0,$$

for every family $(A_{n,\lambda})_{n\in I\!N, \lambda\in\Lambda}$ of subsets of X, such that

$$(o) - \lim_n \left(\sup_\lambda \; \mu(A_{n,\lambda}) \right) = 0.$$

4.16.4.) $(o) - \lim_n \ \sup_{k \geq n} \ \int_{\{x \in X : |f_k(x)| > n\}} \ |f_k| \ d\mu = 0.$

Proof: We observe that we will use convergence in measure only in order to prove the implications [4.16.1.)] $\Longrightarrow$ [4.16.2.)] and [4.16.3.)] $\Longrightarrow$ [4.16.1.)].

[4.16.1.)] $\Longrightarrow$ [4.16.2.)] : See Definition 4.10 and Lemma 4.14.
[4.16.2.)] $\Longrightarrow$ [4.16.3.)] : Fix $\lambda \in \Lambda, \ n, m \in I\!\!N$, with $m \geq n$. By virtue of Lemma 4.15, we have:

$$\int_{A_{n,\lambda}} |f_m| \ d\mu \leq \int_X |f - f_m| \ d\mu +$$

$$+ \int_{A_{n,\lambda}} |f| \ d\mu \leq s_m + e_n \leq s_n + e_n,$$

for some suitable sequences $(s_n)_n$ and $(e_n)_n, \ R \ni s_n \downarrow 0, \ R \ni e_n \downarrow 0$. So,

$$\sup_\lambda \left(\sup_{m \geq n} \int_{A_{n,\lambda}} |f_m| \ d\mu \right) \leq s_n + e_n, \ \forall \ n \in I\!\!N,$$

and therefore

$$(o) - \lim_n \left[\sup_\lambda \left(\sup_{m \geq n} \int_{A_{n,\lambda}} |f_m| \ d\mu \right) \right] = 0.$$

By proceeding analogously, it is easy to prove that

$$\sup_n \int_X f_n \ d\mu \in R.$$

[4.16.3.)] $\Longrightarrow$ [4.16.4.)] : For every $n, k \in I\!\!N$, let $A_{n,k} \equiv \{x \in X : |f_k(x)| > n\}$. Then, there exists $r \in R$, such that

$$r \geq \int_X |f_k| \ d\mu \geq \int_{A_{n,k}} |f_k| \ d\mu \geq \int_{A_{n,k}} n \ d\mu = n \ \mu(A_{n,k}).$$

Thus, $\mu(A_{n,k}) \leq \frac{r}{n}$. So,

$$(o) - \lim_n \int_{A_{n,k}} |f_{n+p}| \ d\mu = 0$$

uniformly with respect to k and $p \in I\!\!N$. Therefore,

$$(o) - \lim_n \left(\sup_{k \geq n} \int_{A_{n,k}} |f_k| \ d\mu \right) = 0.$$

(see also [8]) [4.16.4.)] $\Longrightarrow$ [4.16.3.)] : Let $A_{n,k}$ $(n, k \in I\!N)$ be as in the previous step. For each $n \in I\!N$, and for every $k \in I\!N$, with $k \geq n$, one has:

$$\int_X |f_k| \, d\mu = \int_{X \cap A_{n,k}} |f_k| \, d\mu + \int_{X \cap A^c_{n,k}} |f_k| \, d\mu \leq \alpha_n + n \, \mu(X),$$

where α_n is a suitable decreasing sequence in R, with $\inf_n \ \alpha_n = 0$. Taking $n = 1$, we get:

$$\int_X |f_k| \, d\mu \leq \alpha_1 + \mu(X) :$$

so,

$$\sup_{k \geq 1} \int_X |f_k| \, d\mu \in R.$$

Let now $(E_{n,\lambda})_{n,\lambda}$ be such that $(o) - \lim_n \ \sup_\lambda \ \mu(E_{n,\lambda}) = 0$. Then, $\forall \ \lambda \in \Lambda$, $\forall \ h, n, k \in I\!N$, with $k \geq n$, and $k \geq h$, we have:

$$\int_{E_{n,\lambda}} |f_k| \, d\mu = \int_{E_{n,\lambda} \cap A_{h,k}} |f_k| \, d\mu + \int_{E_{n,\lambda} \cap A^c_{h,k}} |f_k| \, d\mu \leq \rho_h + h \, \mu(E_{n,\lambda}) \leq \rho_h + h \, \sigma_n,$$

for two suitable sequences $(\rho_h)_h$ and σ_n in R, such that $\rho_h \downarrow 0 \downarrow \sigma_n$. Therefore, for every $k \geq n$, we get:

$$\sup_\lambda \left(\sup_{k \geq n} \int_{E_{n,\lambda}} |f_k| \, d\mu \right) \leq \rho_h + h \, \sigma_n.$$

Thus,

$$0 \leq (o) - \limsup_n \left[\sup_\lambda \left(\sup_{k \geq n} \int_{E_{n,\lambda}} |f_k| \, d\mu \right) \right] \leq \rho_h + h \, \inf_n \, \sigma_n = \rho_h, \ \forall \ h.$$

By arbitrariness of h, we get:

$$0 \leq (o) - \limsup_n \left[\sup_\lambda \left(\sup_{k \geq n} \int_{E_{n,\lambda}} |f_k| \, d\mu \right) \right] \leq \inf_h \ \rho_h = 0.$$

Hence,

$$(o) - \lim_n \left[\sup_\lambda \left(\sup_{k \geq n} \int_{E_{n,\lambda}} |f_k| \, d\mu \right) \right] = 0.$$

[4.16.3.)] $\Longrightarrow$ [4.16.1.)] : Fix $\varepsilon > 0$. As $(f_n)_n$ converges in measure to f, then $(f_n)_n$ is Cauchy in measure. So, there exists a sequence

$(z_n)_n,\ z_n \downarrow 0$, such that, for each $n \in I\!N,\ \forall\ m \geq n,\ \mu(A_{n,m}) \leq z_n$, where $A_{n,m} = \{x \in X : |f_n(x) - f_m(x)| > \varepsilon\}$. By 4.16.3.), we have

$$\int_{A_{n,m}} |f_m|\ d\mu \leq t_n,\ \forall\ n \in I\!N,\ \forall\ m \geq n,$$

for a suitable sequence $(t_n)_n,\ t_n \downarrow 0$. Thus, $\forall\ n \in I\!N,\ \forall\ m \geq n :$

$$\begin{aligned} \int_X |f_n - f_m|\ d\mu &= \int_{{A_{n,m}}^c} |f_n - f_m|\ d\mu + \int_{A_{n,m}} |f_n - f_m|\ d\mu \leq \\ &\leq \varepsilon\ \mu(X) + \int_{A_{n,m}} |f_n|\ d\mu + \int_{A_{n,m}} |f_m|\ d\mu \leq \\ &\leq \varepsilon\ \mu(X) + t_n + w_n, \end{aligned}$$

for some suitable sequences $t_n \downarrow 0,\ w_n \downarrow 0$. So, the assertion follows. □

A consequence of Theorem 4.16 is the following:

Corollary 4.17 *With the same hypotheses and notations as above, let $f \in R^X$ be an integrable function. Then there exists an equiintegrable sequence $(f_n)_n$ of simple functions, convergent in measure to f.*

We will prove the following theorem, which is the converse of Corollary 4.17:

Theorem 4.18 *If $f \in R^X$ is such that there exists an equiintegrable sequence $(f_n)_n$ of functions, convergent in measure to f, then f is integrable, and*

$$\int_X f\ d\mu = \lim_n \int_X f_n\ d\mu.$$

Now we compare the integral introduced in 4.11 with the (B)-integral introduced in [3], and the "monotone integral" introduced in [4].

Definition 4.19 Under the same notations as above, a map f is said to be (B)-*integrable* if there exists a sequence $(s_n)_n$ of simple functions, satisfying 4.16.3.) and (o)-convergent in measure to f. In this case, we define

$$(B) - \int_A f\ d\mu \equiv (o) - \lim_n \int_A s_n\ d\mu,\ \forall A \in \mathcal{A}.$$

The following result is a consequence of 4.4 and 4.16.

Theorem 4.20 *Let R be a Dedekind complete Riesz space. Then, every (B)-integrable function f is integrable too. Moreover, if R is Egoroff, f is integrable if and only if it is (B)-integrable.*

Theorem 4.21 *If $f : X \to I\!R$ is bounded measurable, then $\int_X f \, d\mu = (M) - \int_X f \, d\mu$.*

Proof: First of all, we note that the quantity at the right side exists in R, by construction.
Without loss of generality, we may suppose that f is nonnegative.
If f is simple, the assertion is immediate. Now, let $L \equiv \sup_{x \in X} f(x)$, and $(s_n)_n$ be as the functions g_n in Proposition 3.11. of [4]. For every $n \in I\!N$ and $x \in X$, it is:

$$s_n(x) \leq f(x) \leq s_n(x) + \frac{L}{2^n}.$$

So, the sequence $(s_n)_n$ converges uniformly to f. Then

$$(o)\!-\!\lim_n \int_X s_n \, d\mu = (o)\!-\!\lim_n \, (M)\!-\!\int_X s_n \, d\mu = \sup_n \, (M)\!-\!\int_X s_n \, d\mu = (M)\!-\!\int_X f \, d\mu.$$

We observe that the monotone integral satisfies Lemma 4.15 (see also [4]); thus, it follows that $(s_n)_n$ converges in measure to f and satisfies 4.16.3.); so, by Theorem 4.16, we can conclude that f is integrable and $\int_X f \, d\mu = (M) - \int_X f \, d\mu$. □

Theorem 4.22 *Let $f : X \to I\!R$ be a measurable map. Then, the following are equivalent:*

1.) *There exists an equiintegrable sequence of simple functions $(s_n)_n$, convergent in measure to f.*

2.) *f is (M)-integrable.*

3.) *f is integrable.*

Proof. (see also [5]) Without any restriction, we can suppose that f is nonnegative.

[1.)] $\Longrightarrow$ [2.)] : Let $(s_n)_n$ satisfy [1.)]. Then, by Theorem 3.23. of [4] [Vitali 's theorem], it follows that f is integrable, and

$$\int_X f \, d\mu = (o) - \lim_n \int_X s_n \, d\mu = \ (B) - \int_X f \, d\mu.$$

[2.)] $\Longrightarrow$ [3.)] : Assume that f is (nonnegative and) (M)-integrable, and let $(s_n)_n$ be as in Proposition 3.11. of [4]. Then $(s_n)_n$ is an (increasing) sequence of simple functions, convergent in measure to f and satisfying 4.16.3.), because Lemma 4.15 holds for the (M)-integral. Thus, f is integrable.

[3.)] $\Longrightarrow$ [1.)] : Straightforward.

Now, when X is a Banach lattice, it is possible to compare the integral defined in 4.11 with the Bochner integral. The following result holds:

Theorem 4.23 *Let R be a Banach lattice, $\mu : \mathcal{A} \to R$ be an s-bounded finitely additive measure, $f : X \to I\!\!R$ be a map. Then, f is integrable if and only if f is Bochner integrable.*

Proof: We denote by ν a control for μ. If f is integrable, then there exists a sequence $(s_n)_n$ of simple functions, converging in measure to f and uniformly integrable.
Thus, $\int_\cdot f_n \, d\mu \ll m \ll \nu$, uniformly with respect to n. By Theorem 2.5. of [5], f is Bochner integrable.
Conversely, let f be Bochner integrable. Without any restriction, we may assume that f is nonnegative. Then, there exists a sequence $(f_n)_n$ of simple functions, $0 \leq f_n \leq f$, converging in measure to f. Then we have:

$$\int_X f_n \, d\mu \leq \text{ (Bochner)} - \int_X f \, d\mu \ \ll \mu.$$

So, integrability of f follows. □
Hence, in Banach lattices, the Bochner and the monotone integral coincide.

Let now R be a Dedekind complete Riesz space: by Maeda-Ogasawara-Vulikh representation theorem (see also [1]), there exists a compact extremally disconnected topological space Ω such that R can be embedded as a solid subspace of $\mathcal{C}_\infty(\Omega) \equiv \{f : \Omega \to \tilde{I\!\!R} : f$ is continuous, and $\{\omega \in \Omega : |f(\omega)| = +\infty\}$ is nowhere dense in $\Omega\}$.
Now, let $u : [a, b] \to R$ be a Riemann integrable map. Then, there exists a nowhere dense set $N \subset \Omega$ such that the map $t \mapsto u_\omega(t)$, defined by setting $u_\omega(t) \equiv u(t)(\omega)$, is real-valued and bounded. We observe that, for each function $s \in S_u$, for every $\omega \notin N$, the map $s_\omega(t) \equiv s(t)(\omega)$ is a step function, and

$$s(t)(\omega) \leq u(t)(\omega), \ \forall \ t \in [a, b].$$

So we get, up to the complement of a meager set:

$$\left(\int_a^b u(t) \ dt\right)(\omega) = \left[\sup_{s \in S_u} \left(\int_a^b s(t) \ dt\right)\right](\omega) \leq \sup_{w \in S_{u_\omega}} \int_a^b w(t) \, dt = {}_*\int_a^b u(t)(\omega) \, dt;$$

$$\left(\int_a^b u(t) \ dt\right)(\omega) = \left[\inf_{v \in V_u} \left(\int_a^b v(t) \ dt\right)\right](\omega) \geq \inf_{z \in V_{u_\omega}} \int_a^b z(t) \, dt = {}^*\int_a^b u(t)(\omega) \, dt;$$

that is, u_ω is Riemann-integrable, and

$$\left(\int_a^b u(t)\ dt\right)(\omega) = \int_a^b u(t)(\omega)\ dt.$$

Remark 4.24 By proceeding as above, we have that, if $R = \mathcal{B}(D) = \{f \in \mathbb{R}^D : f \text{ is bounded }\}$, where D is an arbitrary set, then

$$\left(\int_a^b u(t)\ dt\right)(d) = \int_a^b u(t)(d)\ dt\ \forall\ d \in D.$$

Endow now D with the discrete topology, let $R^{'} \equiv \mathcal{C}(\beta\ D) = \{f \in \mathbb{R}^{\beta D} : f \text{ is continuous }\}$, and $u : [a, b] \to R^{'}$ be a Riemann-integrable function: then, the map $\xi \mapsto (\int_a^b u(t)\ dt)(\xi)$ is the (unique) continuous extension to the whole of $\beta\ D$ of the map $d \mapsto (\int_a^b u(t)\ dt)(d)$, and thus it is equal to the Chojnacki-integral of the map u, where the chosen retraction $r : \beta\ D \to \beta\ D$ is the identity (see also [2], [7]).

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