A mesh-based notion of differential for TU games

F. Centrone
Dipartimento di Scienze Economiche e Metodi Quantitativi
Università del Piemonte Orientale
Via Perrone 18, 28100 Novara, Italy
francesca.centrone@eco.unipmn.it
Phone+39 0321375320
Fax+39 0321375305

A. Martellotti
Dipartimento di Matematica e Informatica
Università di Perugia
Via Vanvitelli 1, 06123 Perugia, Italy
amart@dmi.unipg.it

Abstract

We introduce the notion of Burkill-Cesari (BC) differentiability for transferable utility (TU) games and compare it with some other analogous established notions existing in cooperative game theoretical literature. We also apply our notion to the study of the core of a new class of TU games.

Key words: TU games, nonadditive set functions, derivatives, Burkill-Cesari differential, Burkill integral.

MSC2010 Classification: 28Axx.

JEL Classification: C71.

1 Introduction

Nonadditive set functions play a fundamental role in game and decision theory: in particular, in the cooperative game-theoretic framework they have been fruitfully used to model the notion of transferable utility (TU) game, that is, given a measurable space \((\Omega, \Sigma)\), a set function \(\nu : \Sigma \rightarrow \mathbb{R}\) such that \(\nu(\emptyset) = 0\) is called a TU game. The set \(\Omega\) and the \(\sigma\)-algebra \(\Sigma\) are interpreted respectively as a set of players and a set of possible coalitions \(S\), while \(\nu(S)\) represents the worth or payoff in terms of “utility” available for division, without restrictions, among the members of \(S\). Though many solution concepts for TU games, i.e., ways of allocating the payoff obtained by a coalition among its players, have been proposed, two of the most famous for the wide range of applications in mathematical economics and social sciences are undoubtedly the Shapley value and the core.
Furthermore, as games with a continuum of players in which no one can affect the final outcome appear naturally wherever there is a “large” number of negligible individuals (like for example consumers in a perfect competition economy), the necessity of infinite dimensional analysis is clear. Since in the finite case the Shapley value of a player represents an average of the “marginal contribution” to the worth of every coalition she can join, in an infinite dimensional context the importance of developing suitable derivative notions is evident in order to generalize this solution concept and treat various classes of games. Differentiation of nonadditive set functions in cooperative game theory dates back to Aumann and Shapley and their fundamental work ([2]). Since then, other notions have been developed by Rosenmuller ([19]), Epstein ([12]), Epstein-Marinacci ([13]), Montrucchio and Semeraro ([18]), and applied mainly to the study of the core of TU games and to some very important class of games, such as for instance the potential ones. All these concepts are very general and, roughly speaking, based on the use of refinements of finite partitions and on the limit made with respect to the partial order induced by them, to capture the idea of increasing smallness of the increment. The common and fundamental idea underlying these notions is that of approximating a generally nonadditive set function (the game) with an additive and hence more treatable one (its derivative). Hence, a measure coincides with its derivative, while nonadditive games in the limit behave almost as additive set functions.

In this paper we introduce a new notion of differentiability for games, called Burkill Cesari (BC) differentiability, where refinements substitute for the classical notion of mesh in order to “measure” the size of a partition. This notion has the advantage, contrarily to what happens with refinements, that it allows to manage “chaotic” families of decompositions of sets. Anyway, the first point which raises when making limits with respect to a mesh, is indeed the choice of a suitable one. If one starts with a “default game” belonging to a special subclass of Aumann and Shapley non atomic games, namely with a monotonic strongly non atomic game, this turns out to induce a “natural” mesh to work with. The property of strong non atomicity is fundamental, and we therefore devote some investigation to the study of its relationships with analogous notions.

Furthermore, we compare our new notion with the Epstein-Marinacci refinement differential ([13]) as well as with the Epstein $\mu$-differentiability ([12]). It is well known that $\mu$-differentiability is stronger than refinement differentiability: under suitable hypotheses we prove that Burkill Cesari differentiability is an intermediate notion. The suitable hypothesis consists in assuming the absolute continuity of $\mu$ w.r.t. the default game in the classical $\varepsilon - \delta$ form (see [4]): various kinds of absolute continuity for games are known in the literature, and we need them throughout the paper in order to prove several results, such as for example the calculus rules. Therefore, we seize this opportunity to devote some space to organic systematization of the several absolute continuity concepts that are most used and scattered in game theoretical literature. For the Burkill Cesari differential we obtain the $\varepsilon - \delta$ absolute continuity w.r.t. the default game in a straightforward way, and this allows to get a representation in terms of an “approximate” Radon-
Nikodym derivative. Under the same hypotheses instead, the refinement differential is absolutely continuous w.r.t the default game in a weaker way, and the same kind of representation as that of the BC differential seems harder to obtain. This will be the subject of further work.

The idea of some “approximate” additivity, makes it natural to investigate the connection between BC differentiability and the concept of Burkill integral for a set function introduced by Cesari ([10]) in the sixties, whose existence is guaranteed for example when one imposes on the set function a condition known as quasi-additivity. Indeed, it turns out that the BC differentiability of a game \( \nu \) at a set \( E \) is equivalent to the Burkill integrability of the increment games \( \nu_E(F) = \nu(E \cup F) - \nu(E) \), with \( F \subset E^c \) and \( \nu_{E^c}(J) = \nu(E) - \nu(E \setminus J) \) with \( J \subset E \). It is worth mentioning the close parallel occurring between the Burkill integral and the refinement derivative of Rosenmuller ([19]) and Montrucchio-Semeraro ([18]), as well as the Epstein-Marinacci differential (just, in one case the limit is made w.r.t. the mesh, while in the other w.r.t. refinements).

To end, we also provide a study and some representation results on the core of Burkill integrable games.

The outline of the paper is as follows: in Section 2 we provide some notation and definitions, in Section 3 we study and compare various notions of nonatomicity for games, in Section 4 we investigate the relationships among different definitions of absolute continuity. Section 5 is devoted to the introduction of the BC differential, its comparison with refinement and \( \mu \)-differentials and to the approximate representation result; in Section 6 we develop calculus rules for the BC differential while Section 7 contains a study of the core of Burkill integrable games.

Throughout, special attention is paid to the discussion of the assumptions and to the comparison with the existing literature. This reverberates through the paper in a large deal of suitable ad hoc examples and counterexamples.

2 Preliminaries

Throughout this work \( \Omega \) will denote a set of players, and \( \Sigma \) the \( \sigma \)-algebra of admissible coalitions. A set function \( \nu : \Sigma \to \mathbb{R} \) such that \( \nu(\emptyset) = 0 \) is called a transferable utility (TU) game.

A game \( \nu \) is

- **positive** if \( \nu(S) \geq 0 \), for all \( S \) in \( \Sigma \);
- **monotone** if for every \( A, B \in \Sigma \) with \( A \subset B \), it holds \( \nu(A) \leq \nu(B) \);
- **superadditive** if for every \( A, B \in \Sigma \) with \( A \cap B = \emptyset \), it holds \( \nu(A \cup B) \geq \nu(A) + \nu(B) \);
- **finitely additive** (or a charge) if \( \nu(S \cup T) = \nu(S) + \nu(T) \), for all pairwise disjoint sets \( S \) and \( T \) in \( \Sigma \).
• countably additive (or a measure) if \( \nu(\bigcup_{n=1}^{\infty} S_n) = \sum_{n=1}^{\infty} \nu(S_n) \) for every countable collection of pairwise disjoint sets \( \{S_n\}_{n=1}^{\infty} \) in \( \Sigma \);

• bounded if \( \sup_{S \in \Sigma} |\nu(S)| < +\infty \).

The set of all finitely additive (resp. countably additive) games bounded w.r.t. the variation norm ([17]) is denoted by FA (resp. CA).

For every \( H \in \Sigma \), we shall denote by \( \Sigma_H \) the \( \sigma \)-algebra defined by \( \Sigma_H := \{ I \in \Sigma : I \subset H \} \).

Given a game \( \nu \) one defines the dual game \( \bar{\nu} \) as

\[
\bar{\nu}(E) = \nu(\Omega) - \nu(E^c).
\]

Note that if \( \nu \) is a monotone game, then \( \bar{\nu} \) remains monotone too.

Given a game \( \nu \), the increment game \( \nu_E : \Sigma_{\Sigma^c} \rightarrow \mathbb{R} \), is defined as \( \nu_E(F) := \nu(E \cup F) - \nu(E) \).

A partition \( D \) of a set \( E \) is a finite family of disjoint elements of \( \Sigma \), whose union is \( E \). By \( \Pi(E) \) we shall denote the set of all the partitions of \( E \).

A partition \( \bar{D} \in \Pi(E) \) is a refinement of another partition \( D \in \Pi(E) \) if each element of \( \bar{D} \) is union of elements of \( D \).

A game \( \nu : \Sigma \rightarrow \mathbb{R} \) is called a scalar measure game if there exists a bounded and convex-ranged charge \( P \), and a real valued continuous function \( g : R(P) \rightarrow \mathbb{R} \), such that \( \nu(\cdot) = (g \circ P)(\cdot) \) on \( \Sigma \), where \( R(P) \) is the range of \( P \).

The core of a game \( \nu \) is the set:

\[
\text{core}(\nu) = \{ \mu \in \text{FA} : \mu(\Omega) = \nu(\Omega) \text{ and } \mu(A) \geq \nu(A) \text{ for all } A \in \Sigma \}.
\]

For all the other terminology we refer the reader to [17].

### 3 Regularity of games

Several different forms and generalizations of the non atomicity concept appear in the literature.

In this section we shall consider the main ones and compare them.

Given a game \( \nu \) we define the following two classes of sets

\[
\mathcal{N}_o(\nu) = \{ E \in \Sigma | \nu(H) = 0 \text{ for every } H \subset E, H \in \Sigma \}
\]

\[
\mathcal{N}(\nu) = \{ N \in \Sigma | \nu(A \cup N) = \nu(A) \text{, for each } A \in \Sigma \}.
\]

Note first that we can alternatively write

\[
\mathcal{N}(\nu) = \{ N \in \Sigma | \nu(A \cap N^c) = \nu(A) \text{, for all } A \in \Sigma \}.
\]

It can be shown that \( \mathcal{N}(\nu) \subset \mathcal{N}_o(\nu) \).

The above inclusion may be strict, as the following example shows
Example 3.1 Let $\Omega = [0,1] \times [0,1]$, $\Sigma = \mathcal{B}_\Omega$ the Borel $\sigma$-algebra, and let $\lambda$ denote the usual Lebesgue measure. Consider the set

$$S = \left\{ \left( \frac{1}{2}, y \right), 0 \leq y \leq 1 \right\}$$

and consider the game $\nu : \Sigma \to \mathbb{R}$ defined as

$$\nu(E) = \begin{cases} 
0 & \text{if } E^o \cap S \neq \emptyset \\
\lambda(E) & \text{otherwise}
\end{cases}$$

Then $\mathcal{N}_o(\nu) = \mathcal{N}_o(\lambda)$.

In fact if $N \in \mathcal{N}_o(\nu)$, then $\nu(N) = 0$. If it were $N^o \cap S = \emptyset$, then necessarily $N^o \neq \emptyset$, and hence $\lambda(N) > 0$; on the other side it cannot be $N^o \setminus S = \emptyset$. Let then $x \in N^o \setminus S$, then $B(x,r) \subset N^o \setminus S$ for a suitable choice of $r$. Set $E = B(x,r)$; as $E \cap S = \emptyset$, by definition $\nu(E) = \lambda(E) > 0$ which in turn implies that $N \not\in \mathcal{N}_o(\nu)$; contradiction. Therefore if $N \not\in \mathcal{N}_o(\nu)$ necessarily $N^o \cap S = \emptyset$, whence $0 = \nu(N) = \lambda(N)$, i.e. $N \in \mathcal{N}(\lambda)$.

Conversely $\mathcal{N}_o(\lambda) \subset \mathcal{N}_o(\nu)$ since $0 \leq \nu \leq \lambda$.

Note that, since $\lambda$ is a measure $\mathcal{N}(\lambda) = \mathcal{N}_o(\lambda)$.

The set $N = (\mathbb{Q} \times \mathbb{Q}) \cap \Omega \in \mathcal{N}(\lambda)$ and therefore $N \in \mathcal{N}_o(\nu)$.

Clearly $N \not\in \mathcal{N}(\nu)$ for if $E = \Omega \setminus N$, since $E^o = \emptyset$, there holds $\nu(E) = \lambda(E) = 1$ while $E \cup N = \Omega$ and $\nu(\Omega) = 0$.

Observe that the game $\nu$ is in the space BV (see [2]); in fact $\nu = \nu_1 - \nu_2$ where

$$\nu_1(E) = \begin{cases} 
\lambda(E) & \text{if } E^o \cap S = \emptyset \\
1 & \text{otherwise}
\end{cases}$$

and

$$\nu_2(E) = \begin{cases} 
0 & \text{if } E^o \cap S = \emptyset \\
1 & \text{otherwise}
\end{cases}$$

are two monotone games.

Definition 3.1 A set $E \in \Sigma$ is an $\mathcal{N}_o$-atom for $\nu$ iff for every $F \in \Sigma_E$ either $F \in \mathcal{N}_o(\nu)$ or $E \setminus F \in \mathcal{N}_o(\nu)$.

Analogously one defines an $\mathcal{N}$-atom for $\nu$.

Definition 3.2 A game $\nu$ is $\mathcal{N}_o$-non atomic if it has no $\mathcal{N}_o$-atoms, $\mathcal{N}$-non atomic if it has no $\mathcal{N}$-atoms. Notice that $\mathcal{N}$ non atomicity is precisely that defined in [2].

A game $\nu$ is strongly non atomic if for every $\varepsilon > 0$ and every $F \in \Sigma$ there exists $D \in \Pi(F)$ with

$$\max_{I \in D} |\nu(I)| < \varepsilon.$$ 

Finally a game is said to be strongly continuous if for every $\varepsilon > 0$ there exists $D \in \Pi(\Omega)$ such that, for every $F \in \Sigma$, $|\nu(I \cap F)| < \varepsilon$ for each $I \in D$. (this last concept is due to F. Ventriglia [20])
We shall compare these four concepts.

It is immediate to recognize that strongly continuous games are strongly non atomic.

Also, since $\mathcal{N}(\nu) \subseteq \mathcal{N}_o(\nu)$, it is clear that $\mathcal{N}_o$-non atomic games are also $\mathcal{N}$-non atomic.

Examples exist in the literature (see [16] or [3]) of non atomic finitely additive measures that are not strongly non atomic; since in the finitely additive case $\mathcal{N}_o(\nu) = \mathcal{N}(\nu)$ these examples show that both $\mathcal{N}_o$- non atomic and $\mathcal{N}$- non atomic games are not necessarily strongly non atomic.

The unanimity game

\[
\nu(E) = \begin{cases} 
1 & \text{for } E = \Omega \\
0 & \text{otherwise}
\end{cases}
\]

is a $\mathcal{N}$- non atomic game for which the grand coalition $\Omega$ is an $\mathcal{N}_o$-atom; it is also a strongly continuous game, for any decomposition finer than $\{\emptyset, \Omega\}$ fulfills the above definition.

In the sequel we shall prove that a strongly non atomic game is necessarily $\mathcal{N}$-non atomic too, and we shall give an example of a strongly non-atomic game that is not strongly continuous.

These two facts will complete the following scheme (a dashed arrow means that the implication is false). $N_oA$ stays for $\mathcal{N}_o$-non atomic games, $NA$ for $\mathcal{N}$-non-atomic, $SNA$ for strongly non-atomic and $SC$ for strongly continuous.

![Diagram]

Example 3.2 Let $f : \Omega \to \mathbb{R}$ be any discontinuous solution of the functional equation

\[
f(x + y) = f(x) + f(y), \; \forall x, y \in \mathbb{R}.
\]

As proven in [6], such a function satisfies the following property:

*for any interval $[a, b] \subseteq \mathbb{R}$ and any $y \in \mathbb{R}$, $f$ attains in $[a, b]$ values arbitrarily close to $y$.*

Let now $\Omega = [0, 1]$ and let $\Sigma$ be the Borel $\sigma$-algebra on $\Omega$. For any set $E \in \Sigma$ let $i_E = \inf E$, $s_E = \sup E$.

Consider the following game $\nu : \Sigma \to [0, +\infty)$

\[
\nu(E) = \begin{cases} 
0 & \text{if } E \text{ is nowhere dense in } \Omega \\
|f[\sup(\overline{E})^o] - f[\inf(\overline{E})^o]| & \text{otherwise.}
\end{cases}
\]

Then $\nu$ is strongly non atomic, but it is not strongly continuous.

To prove that $\nu$ is strongly non atomic, let $E \in \Sigma$ with $\nu(E) > 0$ and $\varepsilon > 0$ be fixed. Since $E$ is not a nowhere dense set, $(\overline{E})^o \neq \emptyset$, and $\inf(\overline{E})^o, \sup(\overline{E})^o$ are cluster points for $(\overline{E})^o$. We shall construct a decomposition $D \in \Pi(E)$ with $\nu(J) \leq 2\varepsilon$ for every $J \in D$. 

6
For the sake of simplicity, let $H = (E)^\circ$, and suppose, to fix ideas, that $f(i_H) < f(s_H)$. Choose now any $x_1 \in H$ with $x_1 > i_H$; since $x_1$ is an interior point, $]x_1 - \delta, x_1 + \delta[ \subset H$ for a suitable choice of $\delta > 0$.

From the above property of $f$ we can choose $\xi_1 \in ]x_1 - \delta, x_1 + \delta[\text{ such that } f(i_H) < f(\xi_1) < f(i_H) + \varepsilon$.

Consider the set $J_1 = [i_H, \xi_1] \cap E$ and note that $\tilde{J}_1 = [i_H, \xi_1] \cap E$; in fact to prove the inclusion $[i_H, \xi_1] \cap E \subset \tilde{J}_1$ (the converse inclusion being obvious) observe that for every point $z \in [i_H, \xi_1] \cap E$, $[i_H, \xi_1]$ is a neighborhood of $z$ and hence contains points from $E$; these points are therefore elements of $J_1$ which shows that $z \in \tilde{J}_1$.

Moreover, $\xi_1 \in ]x_1 - \delta, x_1 + \delta[ \subset \overline{E}$, which easily proves that $\xi_1 \in \tilde{J}_1$.

Now $(\tilde{J}_1)^\circ = [i_H, \xi_1] \cap H \neq \emptyset$, since $\xi_1 \in H$. Thus

$$\nu(J_1) = |f(\sup(\tilde{J}_1)^\circ) - f(\inf(\tilde{J}_1)^\circ)| = f(\xi_1) - f(i_H) < \varepsilon.$$  

Analogously, we can choose a suitable $\xi_2 \in \overline{E}$ with $\xi_2 > \xi_1$ and such that

$$f(i_H) + \varepsilon < f(\xi_2) < f(i_H) + 2\varepsilon$$

and then define $J_2 = [\xi_1, \xi_2] \cap E$; again $(\tilde{J}_2)^\circ = [\xi_1, \xi_2] \cap \overline{E}$ and then

$$\nu(J_2) = f(\xi_2) - f(\xi_1) < f(i_H) + 2\varepsilon - f(i_H) = 2\varepsilon.$$  

Then one iterates this technique, choosing finitely many points $\xi_k$ since $f(i_H) + n\varepsilon$ becomes larger than $f(s_H)$; clearly $D = \{J_1, J_2, \ldots, J_n\}$ is the desired partition of $E$.

On the contrary, $\nu$ is not strongly continuous; in fact if $D \in \Pi(\Omega)$, at least one $J \in D$ is not a nowhere dense set (otherwise $\Omega$ were of first category); but, thanks to the above property of local unboundedness of $f$, every $J \in \Sigma$ that is not a nowhere dense set contains a subset of arbitrarily large $\nu$, and thus, even if $\nu(J) \leq \varepsilon$ for each $J \in D$ this estimate is not hereditarily valid on the subsets.

On the other side, if a game $\nu$ is strongly non atomic, then it is $\mathcal{N}$-non atomic.

To see this, first note that if $E$ is an $\mathcal{N}$ atom for $\nu$ and $F \in \Sigma_E$, then just one between $F$ and $E \setminus F$ is in $\mathcal{N}$; indeed if both $F, E \setminus F$ are in $\mathcal{N}(\nu)$ then both $\nu(F) = \nu(E \setminus F) = 0$ and thus, since $F \in \mathcal{N}(\nu)$,

$$\nu(E) = \nu([E \setminus F] \cup F) = \nu(E \setminus F) = 0$$

while an atom has non zero $\nu$.

This is precisely the difference between $\mathcal{N}_o(\nu)$ and $\mathcal{N}(\nu)$; in $\mathcal{N}_o(\nu)$ such an alternative may be false, as the unanimity game shows.

To reach a contradiction, suppose that $E$ is an $\mathcal{N}$-atom for $\nu$. Since $\mathcal{N}(\nu) = \mathcal{N}(|\nu|)$, it would be an $\mathcal{N}$-atom for $|\nu|$ too. Hence $|\nu|(E) > 0$.

Moreover, by assumption $|\nu|$ is strongly non-atomic.
For each $\varepsilon < |\nu(E)|$ let $D \in \Pi(E)$ be such that $\max\{|\nu(I)|, I \in D\} \leq \varepsilon$. Then by the above property, there should hold $\sharp D > 2$. Actually one finds $\sharp D > n$ for each $n \in \mathbb{N}$ (which is in fact a contradiction); we shall only prove that $\sharp D > 3$, the same reasoning extends to larger integers.

Suppose then that $D = \{I_1, I_2, I_3\}$; then $E = I_1 \cup (I_2 \cup I_3)$, and then just one between $I_1$ and $I_2 \cup I_3$ is in $\mathcal{N}(\nu)$; since it cannot happen $|\nu(I_1)| = |\nu(E)|$, as $|\nu(I_1)| \leq \varepsilon < |\nu(E)|$, necessarily $|\nu(I_2 \cup I_3)| = |\nu(E)|$ and $I_1 \in \mathcal{N}(|\nu|)$.

If both $|\nu(I_2)| > 0, |\nu(I_3)| > 0$, then, since for example $I_3 \not\in \mathcal{N}(|\nu|)$ and $E = (I_1 \cup I_2) \cup I_3$, necessarily $I_1 \cup I_2 \in \mathcal{N}(|\nu|)$; since $I_1 \in \mathcal{N}(|\nu|)$ too

$$|\nu(I_1 \cup I_2)| = |\nu(I_2)| > 0$$

which in turn gives $I_1 \cup I_2 \not\in \mathcal{N}(|\nu|)$ which is absurd. $\square$

The game in Example 3.2 is not a BV game; so the next question that immediately rises is what would the scheme become in the framework of BV games?

What one can remark is that, since the unanimity game is monotone, the comparisons between $N_A$ and $A$, between $SNA$ and $N_oA$, and between $SNA$ and $NA$ remain valid for BV games.

Moreover, something can be said for a special class of BV games, namely those games that for every $\varepsilon > 0$ can be written as the difference of two monotone games, $\nu = \nu' - \nu''$ for which a pair of sets $(N_\varepsilon, P_\varepsilon) \in \Sigma^2$ exists satisfying

- $P_\varepsilon \cap N_\varepsilon = \emptyset$;
- $|\nu(F)| < \varepsilon$ for each $F \subset [\Omega \setminus (N_\varepsilon \cup P_\varepsilon)]$;
- $\nu'(N_\varepsilon) < \varepsilon$, $\nu''(P_\varepsilon) < \varepsilon$.

If such a game is strongly non atomic, then it is necessarily strongly continuous.

In fact, fix $\varepsilon > 0$ and choose the pairs $\nu'_I$, $\nu''_I$ and according $(N_{\frac{\varepsilon}{2}}, P_{\frac{\varepsilon}{2}})$; then, thanks to the strong non atomicity, determine $D_1 \in \Pi(P_{\frac{\varepsilon}{2}}), D_2 \in \Pi(N_{\frac{\varepsilon}{2}})$ such that $\sup_{I \in D_i} |\nu(I)| \leq \frac{\varepsilon}{2}, i = 1, 2$.

Set $D = D_1 \cup D_2 \cup \{\Omega \setminus (N_{\frac{\varepsilon}{2}} \cup P_{\frac{\varepsilon}{2}})\} \in \Pi(\Omega)$; for any $F \in \Sigma$ we shall compute $\nu(I \cap F), I \in D$.

$$|\nu(I \cap F)| \leq \nu'_I(I \cap F) + \nu''_I(I \cap F).$$

For $I \in D_1$ we have that by monotonicity

$$\nu'_I(I \cap F) \leq \nu'_I(I) < \frac{\varepsilon}{2}, \quad \nu''_I(I \cap F) \leq \nu''_I(P_{\frac{\varepsilon}{2}}) < \frac{\varepsilon}{2}.$$ 

Analogously $|\nu(I \cap F)| \leq \varepsilon$ for the $I \in D_2$, and finally

$$|\nu\left(I \cap [\Omega \setminus (P_{\frac{\varepsilon}{2}} \cup N_{\frac{\varepsilon}{2}})\right) < \frac{\varepsilon}{2}$$

according to how the sets have been chosen.
Note that as a particular case, the equivalence between strong non atomicity and strong continuity holds for BV games admitting a sort of Hahn decomposition, namely such that they can be represented as difference of two monotone games \( \nu = \nu' - \nu'' \) that are in a sense mutually singular. More precisely, there exists a pair of sets \((N, P)\) such that \( P \cap N = \emptyset, P \cup N = \Omega \), \( \nu'(N) = \nu''(P) = 0 \); hence the equivalence holds for monotone games, since \((\emptyset, \Omega)\) is a Hahn decomposition with respect to the obvious decomposition \( \nu = \nu - 0 \).

It is worth mentioning that in general \((P, N)\) is not a Hahn decomposition, that is one should not expect that \( \nu \) can be represented as \( \nu(E) = \nu'(E \cap P) - \nu''(E \cap N) \). Indeed we only know that \( N \in \mathcal{N}_0(\nu') \) and \( P \in \mathcal{N}_0(\nu'') \) but we do not know if \( N \in \mathcal{N}(\nu') \) and \( P \in \mathcal{N}(\nu'') \); if this stronger condition did hold, then

\[
\nu'(E) = \nu'(E \cap P), \quad \nu''(E \cap P) = 0
\]

and analogously

\[
\nu''(E) = \nu''(E \cap N), \quad \nu'(E \cap N) = 0
\]

would imply the complete Hahn representation for \( \nu \).

### 4 Absolute continuity for games

Several different extensions of the classical concept of absolute continuity between two measures appeared in the literature. In this section we shall examine the four most used definitions of absolute continuity between two games and we shall compare them.

Given two games \( \mu \) and \( \nu \) we can define the following types of absolute continuity \( \nu \ll \mu \).

Following [18], we shall say that \( \nu \ll_1 \mu \) iff \( \mathcal{N}(\mu) \subset \mathcal{N}(\nu) \).

More classically, we shall say that \( \nu \) is \( \mu \)-absolutely continuous, and write \( \nu \ll_2 \mu \) iff for every \( \varepsilon > 0 \) there exists \( \delta > 0 \) such that when \( |\mu(E)| < \delta \) there also holds \( |\nu(E)| < \varepsilon \).

In [18] the authors introduce the concept of \( \mu \)-continuity of a game, when \( \mu \) is a measure, (but it can be extended to the more general case of \( \mu \) monotone and subadditive). A game \( \nu \) is \( \mu \)-continuous (in symbols \( \nu \ll_3 \mu \)) when \( \nu \) is a continuous map from the pseudometric space \((\Sigma, d_\mu)\), where \( d_\mu \) is the usual Fréchet pseudodistance \( d_\mu(A, B) = \mu(A \Delta B) \). Hence \( \nu \ll_3 \mu \) if for every \( \varepsilon > 0 \) there exists \( \delta = \delta(\varepsilon) > 0 \) such that if \( \mu(A \Delta B) < \delta \) then \( |\nu(A) - \nu(B)| < \varepsilon \).

Finally we shall write \( \nu \ll_4 \mu \) when we shall mean that the absolute continuity by chains holds (see [2]). We recall the definition for the reader’s convenience.

A **chain** is a non-decreasing sequence of sets of the form

\[
\emptyset \subset S_0 \subset S_1 \subset \cdots \subset S_m = \Omega
\]

A **link** of the chain is a pair of successive elements \( \{S_{i-1}, S_i\} \). A **subchain** is a set of links. If \( \mathcal{C} \) is a chain and \( \Lambda \) is a subchain, the **variation of \( \Lambda \) over \( \nu \)** is defined as

\[
\|\nu\|_\Lambda = \sum |\nu(S_i) - \nu(S_i - 1) |
\]
where the sum ranges over all indexes $i$ such that $\{S_{i-1}, S_i\}$ is a link of the subchain $\Lambda$.

Now $\nu \ll_4 \mu$ if for every $\varepsilon > 0$ there exists a $\delta > 0$ such that for every chain $C$ and every subchain $\Lambda$, $\|\mu\|_\Lambda < \delta$ implies $\|\nu\|_\Lambda < \varepsilon$.

We shall now compare these four relationships between $\nu$ and $\mu$.

First it is immediate to get convinced that $\nu \ll_4 \mu$ implies $\nu \ll_2 \mu$; in fact let $|\mu(F)| < \delta(\varepsilon)$ given by the $\ll_4$-continuity, and take the chain $C = \{\emptyset, F\}$.

The implication $\nu \ll_3 \mu \implies \nu \ll_2 \mu$ is immediate, for one takes $B = \emptyset$.

The following proposition compares $\nu \ll_4 \mu$ with $\nu \ll_3 \mu$.

**Proposition 4.1** If $\mu$ is monotone and subadditive, then $\nu \ll_4 \mu \implies \nu \ll_3 \mu$.

**Proof.** Clearly if the game $\mu$ is monotone, then $\mu \geq 0$.

If $\nu \ll_4 \mu$, let $\delta = \delta(\varepsilon)$ be determined by $\nu \ll_4 \mu$; if $0 \leq \mu(A \Delta B) < \delta$ then, by monotonicity, both $\mu(A \setminus B) < \delta$ and $\mu(B \setminus A) < \delta$.

On the other side, each increment game $\nu_E \ll_2 \mu$; in fact, for every $H \subset E^c$ with $\mu(H) < \delta$, we have $0 \leq \mu(E \cup H) - \mu(E) \leq \mu(H) < \delta$; considering the chain $C = \{E, E \cup H\}$ we have that $\|\nu\|_C < \varepsilon$ that is, $|\nu(E \cup H) - \nu(E)| = |\nu_E(H)| < \varepsilon$.

Hence, in particular $\nu_{A \cap B}(A \setminus B) < \varepsilon$, that is $|\nu(A) - \nu(A \cap B)| < \varepsilon$ and analogously $|\nu(B) - \nu(A \cap B)| < \varepsilon$. Hence $|\nu(A) - \nu(B)| < 2\varepsilon$. \qed

Instead we have that $\nu \ll_1 \mu \not\implies \nu \ll_2 \mu$ and $\nu \ll_2 \mu \not\implies \nu \ll_1 \mu$.

To show that $\nu \ll_1 \mu \not\implies \nu \ll_2 \mu$ note that if the implication did hold, it would mean that $0 - \varepsilon$ and $\varepsilon - \delta$ absolute continuities are equivalent, which is well known to be false, even in the framework of finitely additive games; in [7] a counterexample in this sense can be found, namely a pair of finitely additive positive games $\nu$ and $\mu$ such that $\mu(E) = 0 \implies \nu(E) = 0$ (hence $\nu \ll_1 \mu$) but the $\varepsilon - \delta$ absolute continuity fails to be true (see Example 6.1 below). On the other side, to show that $\nu \ll_2 \mu \not\implies \nu \ll_1 \mu$ take $\nu$ to be the unanimity game on $[0, 1]$ endowed with the Borel $\sigma$-algebra, and $\mu$ the Lebesgue measure.

Also $\nu \ll_2 \mu$ does not imply that $\nu \ll_3 \mu$; indeed, if $\nu$ is the unanimity game on $[0, 1]$ and $\mu$ is the usual Lebesgue measure, then $\nu \ll_2 \mu$ for, taking $\delta(\varepsilon) = \frac{1}{2}$ for each $\varepsilon > 0$, every set with $\mu(I) < \delta$ has $\nu(I) = 0$; however $\nu \not\ll_3 \mu$. In fact, $\mu([0, 1] \Delta ([0, 1] \setminus \mathbb{Q})) = 0$, but $\nu([0, 1]) - \nu([0, 1] \setminus \mathbb{Q})) = 1 - 0 = 1$.

Finally we provide an example where $\nu \ll_3 \mu$ but $\nu \not\ll_4 \mu$; this will complete the following scheme for the case of $\mu$ subadditive and monotone:

$$
\nu \ll_4 \mu \implies \nu \ll_3 \mu \implies \nu \ll_2 \mu \not\implies \nu \ll_1 \mu
$$

To this aim, we present first an easy result relative to scalar measure games.
Proposition 4.2 If \( \nu = g \circ P \) is a scalar measure game, then \( \nu \) is monotone iff \( g \) is monotone, and \( \nu \) is strongly non atomic iff \( g \) is continuous at 0.

Proof. We first prove the equivalence for non atomicity. Since \( P \) is convex-ranged it enjoys equivalently the Darboux property ([11]).

\[ \Rightarrow: \] Let \( t_1 < t_2 \in [0, P(\Omega)] \); then by the Darboux Property there exists \( E \in \Sigma \) such that \( P(E) = t_2 \) and \( F \subseteq E \) in \( \Sigma \) with \( P(F) = t_1 \); hence,

\[ g(t_1) = g[P(F)] = \nu(F) \leq \nu(E) = g[P(E)] = g(t_2). \]

The converse implication is straightforward.

We turn now to the second equivalence in the statement.

\[ \Leftarrow: \] Let \( F \in \Sigma \) and \( \epsilon > 0 \) be fixed. Since \( \lim_{x \to 0} g(x) = 0 \), we can find \( n \in \mathbb{N} \) such that \( g \left( \frac{P(F)}{n} \right) < \epsilon \); since \( P \) is convex ranged, it is semiconvex ([7]); thus we can divide \( F \) into \( n \) subsets \( F_1, \ldots, F_n \) with \( P(F_i) = \frac{P(F)}{n} \); so \( \nu(F_i) = g[P(F_i)] < \epsilon \) for each \( i = 1, \ldots, n \).

\[ \Rightarrow: \] Since \( g \) is monotone, we already know that \( \lim_{x \to 0} g(x) \) exists. Let \( \epsilon_n \downarrow 0 \); by the strong non atomicity, we can choose a sequence \( D_n \in \Pi(\Omega) \) such that

- \( \max \{ \nu(I), I \in D_n \} < \epsilon_n \);
- \( D_{n+1} \) refines \( D_n \).

Hence we can start with whichever \( I_1 \in D_1 \) and then choose iteratively \( I_n \in D_n \) so that the sequence \( I_n \) is decreasing; thus on one side \( \nu(I_n) < \epsilon_n \) and hence \( \nu(I_n) \downarrow 0 \). But \( P(I_n) \) is decreasing too, and so it converges to some \( t_o \geq 0 \); by the monotonicity of \( g \),

\[ g(t_o) \leq \lim g[P(I_n)] = \lim \nu(I_n) = 0; \]

if \( t_o = 0 \) this is enough to prove the required continuity at 0. If \( t_o > 0 \) by monotonicity, \( g = 0 \) in \( [0, t_o] \) which again yields the required continuity.

We are now ready for the promised example.

Example 4.1 Let \( (\Omega, \Sigma) = ([0,1], B) \) and let \( f : [0,1] \to [0,1] \) be the classical Vitali-Cantor ternary function; as it is well known \( f \) is continuous and non decreasing, but it is not absolutely continuous.

Then if \( \lambda \) denotes the usual Lebesgue measure, and \( \nu = f \circ \lambda \), \( \nu \) is a monotone game with \( \nu \ll_3 \lambda \) but \( \nu \not\ll_4 \lambda \).

For every \( A, B \in \Sigma \) with \( \lambda(A \Delta B) < \delta(\epsilon) \), where \( \delta \) is determined by the uniform continuity of \( f \), there follows from \( |\lambda(A) - \lambda(B)| \leq \lambda(A \Delta B) \) that

\[ |\nu(A) - \nu(B)| = |f[\lambda(A)] - f[\lambda(B)]| \leq \epsilon \]
which proves that \( \nu \ll_3 \lambda \).

On the other side, \( f \) is not absolutely continuous; therefore there exists \( \varepsilon > 0 \) such that for each \( \delta > 0 \) there exists a finite union of non-overlapping intervals

\[
E_\delta = \bigcup_{i=1}^{n_\delta} (x_i^\delta, y_i^\delta)
\]

with \( \lambda(E_\delta) = \sum_{i=1}^{n_\delta} (y_i^\delta - x_i^\delta) < \delta \) but \( \sum_{i=1}^{n_\delta} [f(y_i^\delta) - f(-x_i^\delta)] > \varepsilon. \)

Moreover, we can always suppose that \( x_1^\delta = 0. \)

Fix any \( \delta > 0; \) we shall omit the \( \delta \) sub-super scripts for the sake of simplicity.

Consider then the following chain \( C = \{S_1, \ldots, S_{2n-1}\}: \)

\[
S_1 = \{0\}, \ S_2 = (0, y_1), \ S_3 = (0, x_2), \ S_4 = (0, y_2), \ldots, S_{2n-1} = (0, x_n), \ S_{2n} = (0, y_n).
\]

Consider the subchain \( \Lambda = \{(1, 2), (3, 4), \ldots, (2n-1, 2n)\}; \) then

\[
\|\lambda\|_\Lambda = \sum_{i=1}^{n_\delta} (y_i^\delta - x_i^\delta) < \delta
\]

while

\[
\nu(S_k) = f[\lambda(S_k)] = f[\lambda((0, \xi))] = f(\xi)
\]

whence

\[
\|\nu\|_\Lambda = [\nu(S_1) - \nu(S_0)] + [\nu(S_3) - \nu(S_2)] + \ldots + [\nu(S_{2n-1}) - \nu(S_{2n-2})] = \\
= f(y_1) + [f(y_2) - f(x_2)] + \ldots + [f(y_n) - f(x_n)] > \varepsilon
\]

which proves that \( \nu \not\ll_4 \lambda. \)

**Remark 4.1** Having introduced the class \( \mathcal{N}_\nu(\nu) \) in the previous section, it is rather natural to ask what further kind of absolute continuity between games it would introduce. In this remark we shall briefly discuss this topic. For the sake of convenience we shall use the symbol \( \nu \ll_o \mu \) when \( \mathcal{N}_\nu(\mu) \subset \mathcal{N}_\nu(\nu); \) the reason why we shall never use it, is that, apart for the case of countably additive measures, where \( \ll_o \) coincides with each of the other four forms, for more general games this form of \( \mu \) regularity of \( \nu \) is never considered in the literature. The reason is probably that in general it does not compare to the other forms of \( \mu \) regularity of the game \( \nu. \)

When \( \mu \) and \( \nu \) are finitely additive monotone games, it is evident that \( \nu \ll_o \mu \) is equivalent to \( \nu \ll_1 \mu \) and that \( \nu \ll_2 \mu, \nu \ll_3 \mu, \nu \ll_4 \mu \) are all equivalent. Therefore any pair of non-negative finitely additive measures for which \( \varepsilon - \delta \) absolute continuity does not imply \( \varepsilon - \delta \) absolute continuity (for instance Example 3.7 in [7] - see Example 6.1 below) provides a counterexample for any of the implications \( \nu \ll_o \mu \not\iff \nu \ll_i \mu \), \( i = 2, 3, 4. \)

Hence the interesting comparison is between \( \ll_o \) and \( \ll_1. \)
Since in **Example 3.1** we have that
\[ N(\nu) \subseteq N_o(\nu) = N_o(\lambda) = N(\lambda) \]
it is clear that \( \lambda \ll_o \nu \) but \( \lambda \ll_1 \nu \).

For the converse implication, let the game \( \mu \) be given by
\[
\mu(E) = \begin{cases} 
0 & \text{if } E \in N(\nu) \\
1 & \text{if } E \notin N(\nu)
\end{cases}
\]
where \( \nu \) is the same as in **Example 3.1**.

We have that if \( E \in N(\nu) \), and \( F \subset E \) then \( F \in N(\nu) \) too. Therefore \( \mu \) is monotone. Also, if \( E, F \in N(\nu) \) then \( E \cup F \in N(\nu) \) too, hence one easily checks that \( \mu \) is subadditive. Therefore \( N(\mu) = N_o(\mu) \).

On the other side immediately \( N_o(\mu) = N(\nu) \), thus
\[ N_o(\mu) = N(\mu) = N(\nu) \subseteq N_o(\nu) \]
whence \( \nu \ll_1 \mu \) but \( \nu \ll_o \mu \).

## 5 The Burkill-Cesari differential of games

In this section we shall introduce a new definition of differential for games: to this aim, the first concept we shall need is that of a mesh on a measurable space.

Let \( (\Omega, \Sigma) \) be a measurable space. Similarly to [10], we shall define a *mesh* on \( \Omega \) in the following way:

**Definition 5.1** Let \( \mathcal{A} \) be any subclass of \( \Sigma \), and let \( \mathcal{D} = \{ D \in \Pi(F), F \in \mathcal{A} \} \).

A map \( \delta : \mathcal{D} \to [0, +\infty) \) is a *mesh*, provided the following properties hold:

1. (d.1) for every \( G \in \mathcal{A} \) and every \( D \in \Pi(G) \), \( \delta(D) > 0 \);
2. (d.2) for every \( \varepsilon > 0 \) and \( G \in \mathcal{A} \) there are systems \( D \in \Pi(G) \) with \( \delta(D) < \varepsilon \);
3. (d.3) if \( D_2 \in \Pi(E) \) refines \( D_1 \), then \( \delta(D_2) \leq \delta(D_1) \);
4. (d.4) for every \( G \in \mathcal{A} \), and every \( D_o \in \Pi(G) \), say \( D_o = \{ G_1, \ldots, G_n \} \), and for every choice of \( D_i \in \Pi(G_i), i = 1, \ldots, n \), setting \( D = D_1 \cup \ldots \cup D_n \) we have \( \delta(D) = \max_{1 \leq i \leq n} \delta(D_i) \).

Let now \( \nu \) be a monotone strongly non-atomic game on \( (\Omega, \Sigma) \); then if \( \mathcal{A} = \Sigma^+ = \Sigma \setminus N_o(\nu) \), we can define \( \delta_\nu \) as
\[
\delta_\nu(D) = \max\{ \nu(I), I \in D \}. \tag{1}
\]

If \( \nu \) is not the null game, then \( \delta_\nu \) is a mesh according to the above definition.
Conversely, when a mesh is defined on \( \mathcal{S} \), and \( \emptyset \notin \mathcal{S} \), then the game

\[
\nu(E) = \begin{cases} 
0 & \text{if } E \notin \mathcal{S} \\
\delta(\{E\}) & \text{if } E \in \mathcal{S}
\end{cases}
\]

is monotone, non atomic, and \( \delta = \delta_\nu \).

In fact, by (d.1), \( \nu(E) \geq 0 \); if \( E \subset F \) and they both are in \( \Sigma^+ \), then by (d.3)

\[
\delta(\{E, F \setminus E\}) \leq \delta(\{F\}) = \nu(F)
\]

and by (d.4),

\[
\nu(E) = \delta(\{E\}) \leq \delta(\{E, F \setminus E\}) \leq \delta(\{F\}) = \nu(F).
\]

The inequality \( \nu(E) \leq \nu(F) \) is trivial if \( E \) and/or \( F \) are in \( \mathcal{N}_0(\nu) \).

To prove that \( \nu \) is strongly non atomic, let \( F \in \Sigma^+, \varepsilon > 0 \) be fixed. By (d.2) there exists \( D \in \Pi(F) \) with \( \delta(D) < \varepsilon \). By (d.4), if \( D = \{I_1, \ldots, I_n\} \), and we set \( D_k = \{I_k\} \in \Pi(I_k) \), then \( D = D_1 \cup \ldots \cup D_n \) and \( \delta(D_k) \leq \delta(D) < \varepsilon \) that is \( \nu(I_k) = \delta(D_k) < \varepsilon \).

Finally (d.4) shows that \( \delta = \delta_\nu \).

From now on in this section we shall consider a strongly non atomic monotone game \( \lambda \) on \((\Omega, \Sigma)\) and the mesh \( \delta \) to which we shall refer will always be the default mesh \( \delta_\lambda \) defined by (1).

Note that if \( \nu \) is a monotone game, then any increment game remains monotone too.

In the literature several definitions of derivative or differential of a game appeared (see [13], [18]), all dealing with the behavior of finite sums over partitions, when the “size” of the summands decreases.

Here we propose an alternative formulation, inspired by the Burkill-Cesari integrability introduced in the sixties in [10].

**Definition 5.2** Let \( \nu \) be a game on \((\Omega, \Sigma)\) and let \( E \in \Sigma \) be fixed; we shall say that \( \nu \) has an **outer Burkill-Cesari differential** (briefly **outer BC-differential**) if there exists a strongly non atomic finitely additive measure \( \partial^+_{E^c}(\nu, \cdot) \) defined on \( \Sigma_{E^c} \), and such that for every \( F \subset E^c \)

\[
\lim_{\delta(D) \to 0} \sum_{I \in D} |\nu_E(I) - \partial^+_{E^c}(\nu, I)| = 0;
\]

similarly we shall say that \( \nu \) admits an **inner BC differential** \( \partial^-_{E^c} \) defined on \( \Sigma_E \), if, for every \( G \subset E \)

\[
\lim_{\delta(D) \to 0} \sum_{I \in D} |\bar{\nu}_{E^c}(I) - \partial^-_{E^c}(\nu, I)| = 0.
\]

Finally, if \( \nu \) admits both the outer and the inner BC differentials at \( E \), we shall define \( \partial_E : \Sigma \to \mathbb{R} \) as \( \partial_E(\nu, I) := \partial^+_{E^c}(\nu, I \cap E^c) + \partial^-_{E^c}(\nu, I \cap E) \) as the **BC differential** of \( \nu \) at \( E \).
More generally, if a set function \( \phi : \Sigma \to \mathbb{R} \) has \( \phi(\emptyset) \neq 0 \), one considers the game \( \nu(\cdot) = \phi(\cdot) - \phi(\emptyset) \).

Then for every \( E \in \Sigma \) the increments coincide; therefore we can extend the notion of outer (resp. inner) BC differentials to set functions, for one would find \( \nu_E = \phi_E \) and \( \bar{\nu}_{E^c} = \bar{\phi}_{E^c} \); hence one would find \( \partial^+_E(\nu, \cdot) = \partial^+_E(\nu, \cdot) \) and \( \partial^-_E(\nu, \cdot) = \partial^-_E(\nu, \cdot) \).

Note moreover that, for every \( J \subset E \)

\[
\bar{\nu}_{E^c}(J) = \bar{\nu}(E^c \cup J) - \bar{\nu}(E^c) = \nu(\Omega) - \nu((E^c \cup J)^c) - \nu(\Omega) + \nu(E) = \nu(E) - \nu(E \setminus J)
\]

which gives to the definition of \( \partial^-_E(\nu, \cdot) \) above a formulation more similar to the inner refinement differential ([13]).

Also, if the BC differential at a set exists, then it is unique.

**Remark 5.1** In [10] the Burkill-Cesari integral of a game \( \nu \) is defined as

\[
E \mapsto \int_E \nu = \lim_{\delta(D) \to 0} \sum_{I \in D} \nu(I).
\]

We report the following result

**Proposition 5.1** Let \( \nu \) be Burkill integrable. Then \( \int \nu \) is finitely additive.

**Proof.** See [1] (Theorem II): there \( \Omega \) is assumed to be a topological space, and only special decompositions are taken into account, but it can be checked that the same proof applies to our setting.

In complete analogy with Theorem 3.1 in [18], it can be shown that the limits in (2) and (3) are equivalent to

\[
\lim_{\delta(D) \to 0} \sum_{I \in D} \nu_E(I) = \partial^+_E(\nu, F); \quad (5)
\]

and

\[
\lim_{\delta(D) \to 0} \sum_{I \in D} \bar{\nu}_{E^c}(I) = \partial^-_E(\nu, G). \quad (6)
\]

or else that \( \int_F \nu_E = \partial^+_E(\nu, F), F \subset E^c \) and \( \int_G \bar{\nu}_{E^c} = \partial^-_E(\nu, G), G \subset E \).

To be more precise, what one could prove is that if the increment games admit Burkill-Cesari integrals, then these coincide with the additive set functions in (2) and (3), and if they are strongly non atomic, then they coincide with the BC outer and inner differentials.

Next, we show that in particular significant cases, the BC differential is independent on the default game.
Proposition 5.2 Let \( \nu \) be a game BC differentiable at some \( E \in \Sigma \) and let \( \mu \) be a strongly non-atomic monotone game such that \( \lambda \ll_2 \mu \). Then \( \nu \) is Burkill-Cesari differentiable at \( E \) with the respect to the alternative mesh induced by \( \mu \), and \( \partial E^\lambda(\nu, \cdot) = \partial E^\mu(\nu, \cdot) \).

Proof. Fix \( \varepsilon > 0 \) and let \( \eta(\varepsilon) \) by the \( \ll_2 \) absolute continuity and \( \eta_1 = \eta_1(\varepsilon) \) by the \( \delta_\lambda \)- BC differentiability of \( \nu \). We shall prove the result for the outer differential. Fix \( F \subset E^c \) and let \( D \in \Pi(F) \) have \( \delta_\mu(D) < \eta_1(\varepsilon) \). Then each \( I \in D \) has \( \mu(I) < \eta_1(\varepsilon) \) and so \( \sum_{I \in D} |\nu(I) - \partial E^\lambda(\nu, I)| < \varepsilon \) which completes the proof. \( \square \)

We shall now compare this new notion with the refinement differentials of Epstein and Marinacci ([13]). Recall that these are defined analogously to the BC differentials: one simply replaces in (2) and (3) the limit w.r.t. the mesh \( \delta_\lambda \) with the limit with respect to refinements of partitions.

It is immediate to recognize that if the game \( \nu \) admits inner (resp. outer) BC differential, then \( \partial E^+(\nu, \cdot) \) (resp. \( \partial E^-(\nu, \cdot) \)) are also the refinement differentials \( \delta E^+(\nu, \cdot), \delta E^-(\nu, \cdot) \) of [13] (Montrucchio and Semeraro [18] do not require the strong non atomicity of the differentials); the converse implication is not true, as a following example will show.

Example 5.1 Let \( \Omega \) and \( \Sigma = B_{\Omega} \) be as in Example 3.1, and consider the game \( \nu : \Sigma \to \mathbb{R} \) defined as
\[
\nu(E) = \begin{cases} \sqrt{\lambda(E)} & \text{if } E^o \cap S \neq \emptyset \\ \lambda(E) & \text{otherwise} \end{cases}
\]
Then \( \nu \) is a monotone strongly non atomic game.

Let \( E \in \Sigma \) be fixed; if \( E^o \cap S = \emptyset \), for every \( D \in \Pi(E) \) it is
\[
\sum_{I \in D} \nu(I) = \lambda(E)
\]
and hence the refinement limit, as well as the \( \delta_\lambda \)-limit exist. Hence \( \delta_0^+(\nu, E) = \lambda(E) \) in both senses.

If \( E^o \cap S \neq \emptyset \) however, for every decomposition \( D \in \Pi(E) \) we can always find a refinement \( D' \) of \( D \) such that \( I^o \cap S = \emptyset \) for every \( I \in D' \), and hence again
\[
\sum_{I \in D'} \nu(I) = \lambda(E).
\]
Thus also in this case the refinement limit exists and coincides with \( \lambda(E) \).

However \( \nu \) does not admit BC-differential at \( \emptyset \); indeed take \( E = \Omega, \varepsilon > 0 \) and let us consider \( n \in \mathbb{N} \) such that \( \frac{1}{n} < \varepsilon \); consider then the partition \( D = \{I_1, \ldots, I_n\} \in \Pi(\Omega) \) where
\[
I_k = [0, 1] \times \left[ \frac{k - 1}{n}, \frac{k}{n} \right].
\]
Then \( \lambda(I_k) = \frac{1}{n} < \varepsilon \) for each \( k \) and so \( \delta(D) < \varepsilon \), but
\[
\sum_{I \in D} [\nu(I) - \lambda(I)] = n \nu(I_k) - 1 = \sqrt{n} - 1 \to +\infty.
\]
Thus

\[
\limsup_{\delta(D) \to 0} \sum_{I \in D} |\nu(I) - \lambda(I)| = +\infty.
\]

Next we shall compare this new concept with the \(\mu\)-differentiability of [12]. Actually we shall prove that the BC differentiability is implied by a weaker form of \(\mu\)-differentiability.

**Definition 5.3** Let \(\mu\) be a strongly non atomic non-negative finitely additive measure on \(\Sigma\). A game \(\nu\) is called \(\mu\)-differentiable at \(E \in \Sigma\) if

(i) \(\nu \ll_1 \mu\);

(ii) there exists a strongly non atomic finitely additive measure \(\Delta_E(\nu, \cdot)\) on \(\Sigma\) such that

\[
\lim_{\mu(F \cup G) \to 0} \frac{|\nu(E \cup F \setminus G) - \nu(E) - \Delta_E(\nu, F) + \Delta_E(\nu, G)|}{\mu(F \cup G)} = 0
\]

with \(F \subset E^c\) and \(G \subset E\), and \(\mu(F \cup G) > 0\).

Epstein [12] assumes the following equivalent form of (i):

for every \(N, N' \in \Sigma\) such that \(\mu(N \Delta N') = 0\), then \(\nu(E \cup N) = \nu(E \cup N')\) for every \(E \in \Sigma\). This condition is equivalent to \(\nu \ll_1 \mu\) since, if \(\mu(N \Delta N') = 0\), both \(N \setminus N'\) and \(N' \setminus N\) are in \(\mathcal{N}(\mu)\); then if \(\nu \ll_1 \mu\) they are also in \(\mathcal{N}(\nu)\); thus for every \(A \in \Sigma\)

\[
\nu(A \cup N) = \nu[A \cup (N \cap N') \cup (N \setminus N')] = \nu[A \cup (N \cap N') \cup (N' \setminus N)] = \nu(A \cup N').
\]

Viceversa, for every \(N \in \mathcal{N}(\mu)\) it is \(\mu(N \Delta \emptyset) = 0\); so if Epstein’s condition holds \(\nu(A \cup N) = \nu(A \cup \emptyset) = \nu(A)\) for every \(A \in \Sigma\) that is, \(N \in \mathcal{N}(\nu)\).

We shall now compare this last differentiability, that according to [12] also implies the refinement differentiability, with the BC differentiability.

**Theorem 5.1** Let \(\mu\) be as in Definition 5.3, and suppose that \(\mu \ll_2 \lambda\). Let \(\nu\) be a game \(\mu\)-differentiable at \(E \in \Sigma\). Then \(\nu\) admits BC differential at \(E\) and \(\partial_E(\nu, \cdot) = \Delta_E(\nu, \cdot)\).

**Proof.** As stated in [12], \(\Delta_E(\nu, \cdot) \ll_1 \mu\).

On the other side if \(\mu(F) = 0\), \(F \in \mathcal{N}(\mu) \subset \mathcal{N}(\nu)\) and so \(\nu_E(F) = 0 < \varepsilon\).

Analogously one could prove this implication for \(\nu_{E^c}\).

We shall now prove that for every \(F \subset E^c\) there holds

\[
\lim_{\delta \mu(D) \to 0} \sum_{I \in \delta \mu(D)} |\nu_E(I) - \Delta_E(\nu, I)| = 0.
\]
Let \( \rho = \rho(\varepsilon) \) be determined by the \( \mu \)-differentiability. Then for every \( I \subset F \) with \( 0 \leq \mu(I) < \rho \) we have

\[
|\nu_E(I) - \Delta_E(\nu, I)| \leq \varepsilon \mu(I)
\]

Note that this also holds when \( \mu(I) = 0 \) as \( \nu \ll \mu \) and \( \Delta_E(\nu, \cdot) \ll \mu \).

Now let \( D \in \Pi(F) \) with \( \delta_\mu(D) < \rho(\varepsilon) \). Then

\[
\sum_{I \in D} |\nu_E(I) - \Delta_E(\nu, I)| \leq \varepsilon \mu(F).
\]

Let \( \eta = \eta[\rho(\varepsilon)] \) be determined by the absolute continuity \( \mu \ll_2 \lambda \). Fix \( \varepsilon > 0 \) and \( F \subset E^c \) and let \( D \in \Pi(F) \) have \( \delta(D) < \eta[\rho(\varepsilon)] \). Then \( \delta_\mu(D) < \rho(\varepsilon) \) and hence \( \sum_{I \in D} |\nu_E(I) - \Delta_E(\nu, I)| \leq \varepsilon \mu(F) \) that is

\[
\lim_{\delta(D) \to 0} \sum_{I \in D} |\nu_E(I) - \Delta_E(\nu, I)| = 0.
\]

The inner differential can be analogously treated.

So far we have shown that the BC differential is an intermediate concept between the Epstein-Marinacci refinement differential and the Epstein one.

In the sequel of this section we shall investigate which properties of the \( \mu \)-differential are preserved by the BC differential.

First we shall prove a version of Lemma 3 in [9]. To this aim, we are interested in the cases when the differentials are absolutely continuous with respect to the default game \( \lambda \); in other words one would control the size of \( \partial E^\pm \) on sets of small \( \lambda \) size, which may happen to have large increment size.

To discuss this topic we shall remind some concepts, relative to the Burkill-Cesari integral.

In [1] the authors define the \( \delta_\lambda \)-continuity of a game \( \nu \); in our setting this precisely coincides with the condition \( \nu \ll_2 \lambda \). In particular Theorem V in [1], whose proof does not make use of the topological structure there assumed, becomes

**Proposition 5.3** If \( \nu \) is Burkill-Cesari integrable, then \( \int \nu \ll_2 \lambda \) iff \( \nu \ll_2 \lambda \).

Hence, if the game \( \nu \) admits outer (resp. inner) BC differential at \( E \), then \( \partial_+ E \ll_2 \lambda \) iff \( \nu_E \ll_2 \lambda \) (and analogously for the inner differential).

Moreover if both the BC outer and inner differentials exist at \( E \), then \( \partial E \ll_2 \nu \) iff \( \nu_E \ll_2 \lambda \). Hence we are induced to focus on the cases when the increment games \( \nu_E \ll_2 \lambda \) and/or \( \bar{\nu}_E \ll_2 \lambda \).

Unfortunately, in general if \( \nu \ll_2 \lambda \) does not imply that \( \nu_E \ll_2 \lambda \); for instance if \( \nu \) is the unanimity game on \([0, 1]\) with the usual Lebesgue measure \( \lambda \) as the default game, and \( E = [0, 1] \setminus \mathbb{Q} \), then \( \nu \ll_2 \lambda \) but \( \nu_E \nolll_2 \lambda \), because, if \( \varepsilon < 1 \), then \( \nu(E^c) = 0 < \delta \) but \( \nu_E(E^c) = \nu(E \cup E^c) - \nu(E) = \nu(\Omega) = 1 > \varepsilon \).

However, in the case of measure games something more can be said
Proposition 5.4 Let $\nu = g \circ P$ with $g$ continuous, and $P$ countably additive; if $P \ll_2 \lambda$, then $\nu \ll_2 \lambda$ and the increment games $\nu_E, \nu_E^-$ are always $\ll_2 \lambda$.

Proof. To prove this statement, note that $g$ is uniformly continuous on the range $R(P)$ (for it is compact, according to Lyapunouov Theorem). Let $\delta = \delta(\varepsilon)$ be determined by the uniform continuity of $g$, and let $\rho = \rho(\tau)$ be determined by the absolute continuity $P \ll_2 \lambda$; then if $0 \leq x \leq \rho(\tau)$, we have $g(x) < \tau$.

Let $H$ have $\lambda(H) < \delta(\rho(\varepsilon))$; then $P(H) < \rho(\varepsilon)$ and hence $\nu(H) = g[P(H)] < \varepsilon$, namely $\nu \ll_2 \lambda$.

Moreover, using the continuity at every $E \in \Sigma$, if $H \subset E^c$

$$\nu_E(H) = g[P(E \cup H)] - g[P(E)] = g[P(E) + P(H)] - g[P(E)] < \varepsilon$$

and if $H \subset E$ then

$$\nu_E^-(J) = \nu(E) - \nu(E \setminus H) = g[P(E)] - g[P(E) - P(H)] < \varepsilon.$$

One could alternatively require that $P$ is simply finitely additive, provided $g$ is assumed to be continuous on the closure of $R(P)$ (which is, in fact, compact - see [7]).

For more general games we need a stronger absolute continuity with respect to the default game.

Theorem 5.2 Let $\lambda$ be a strongly non atomic finitely additive game, and let $\nu$ admit BC differential at some $E \in \Sigma$; if $\nu \ll_3 \lambda$, then $\partial_E(\nu, \cdot) \ll_2 \lambda$.

Proof. First of all note that $\nu \ll_3 \lambda$ implies $\nu_E \ll_2 \lambda$ and $\nu_E^- \ll_2 \lambda$. Indeed by taking $A = E \cup I$ and $B = E$, it is $A \Delta B = I$ and hence, from $\nu \ll_3 \lambda$ and fixed $\varepsilon > 0$, there exists $\delta(\varepsilon) > 0$ such that $\lambda(I) < \delta$ implies $|\nu_E(I) - \nu(E)| < \varepsilon$, that is $\nu_E \ll_2 \lambda$. An analogous argument applies to $\nu_E^-$. From Proposition 5.3 then we have

$$\partial_E^+(\nu, \cdot) = (\delta_\lambda - \text{BC}) \int \nu_E \ll_2 \lambda.$$

Repeating the same argument for the inner BC differential, we reach $\partial_E(\nu, \cdot) \ll_2 \lambda$. \qed

This absolute continuity of the BC differential with respect to the default game will be of crucial importance for the results in the next section.

It is important to mention that in the case of refinement differentials, all what can be said is that $\delta_E(\nu, \cdot) \ll_1 \lambda$. Indeed, by Proposition 3.2.iv) in [18], one gets $\delta_E(\nu, \cdot) \ll_1 \nu$, that is $\mathcal{N}(\nu) \subset \mathcal{N}[\delta_E(\nu, \cdot)]$; on the other side, assumption $\nu \ll_4 \lambda$ implies $\nu \ll_1 \lambda$ and hence $\mathcal{N}(\lambda) \subset \mathcal{N}(\nu) \subset \mathcal{N}[\delta_E(\nu, \cdot)]$. However, if we do not assume a priori that $\delta_E(\nu, \cdot)$ is countably additive, the $\ll_1$ absolute continuity may fail to imply the $\varepsilon - \delta$ absolute continuity.
We have not been able so far to find a suitable counterexample: however, the fact that the BC differential surely enjoys this stronger absolute continuity makes it more convenient than the refinement differential under many respects.

When \( \lambda \) is a measure, \( \partial_E(\nu, \cdot) \) is also a countably additive measure, which we can represent by means of a density

\[
\partial_E(\nu, F)N = \int_F f d\lambda, \ F \in \Sigma.
\]

The next result will provide a way to compute the Radon-Nikodym derivative \( f \) directly by means of the game \( \nu \).

To this aim, given \( F \in \Sigma \), and \( D \in \Pi(F) \) we define the projection of a game \( \nu \ll_1 \lambda \) on \( D \) as

\[
\nu_D = \sum_{I \in D} \frac{\nu(I)}{\lambda(I)} 1_I
\]

(where \( \begin{pmatrix} 0 \\ 0 = 0 \end{pmatrix} \)).

Let us assume now that \( \lambda \) is a measure, and that \( \nu \ll_3 \lambda \); we fix then a sequence of partitions of \( \Omega \) in the following way: \( D_0 = \{ E, E^c \} \), and for every \( n \in \mathbb{N}^+ \) \( D_n \) is a refinement of \( D_{n-1} \) obtained by ”halving” each set in the \( D_{n-1} \)-th decomposition, namely, if \( D_n = \{ I_1, \ldots, I_{2n+1} \} \) then we choose for each set \( I_k \) a decomposition into two disjoint sets, \( I = J \cup H \), each having \( \lambda(J) = \lambda(H) = \frac{\lambda(I)}{2} \).

Then we define for every \( n \in \mathbb{N} \) the maps

\[
\varphi_n^E = (\nu_E)_{D_n}, \quad \varphi_n^{E^c} = (\nu_{E^c})_{D_n}, \quad \psi_n^E = \varphi_n^E 1_{E^c} + \varphi_n^{E^c} 1_E.
\]

**Theorem 5.3** If \( \lambda \) is a measure, \( \nu \ll_3 \lambda \) and \( \nu \) admits BC differential at \( E \in \Sigma \); then

\[
\partial_E(\nu, F) = \lim_{n \to +\infty} \int_F \psi_n^E d\lambda
\]

for every \( F \in \Sigma \).

**Proof.** As it is well known (see, for instance [5]), setting \( f_D = [\partial_E(\nu, \cdot)]_D \) and \( F_D = \int f_D d\lambda \) one has \( \lim_D F_D = \partial_E(\nu, \cdot) \) in the variation of \( ca(\Sigma) \) (and where the limit is meant in the sense of refinements). Hence, setwise \( \text{var}[F_D - \partial_E(\nu, \cdot)](F) \to 0 \) that is, since \( \text{var}[F_D - \partial_E(\nu, \cdot)](F) = \int \| f_D - f \| d\lambda \), the limit

\[
\lim_{D \in \Pi(F)} \int_F \| f_D - f \| d\lambda = 0
\]

for every \( F \in \Sigma \). Moreover, as shown by Leader ([15], Theorem 2), the map \( D \mapsto \int_F \| f_D - f \| d\lambda \) is non-decreasing with respect to refinements on \( \Pi(F) \). Hence

\[
\lim_{n \to +\infty} \int_F \| f_{D_n} - f \| d\lambda = 0
\]
holds for every $F \in \Sigma$, from where
\[ \partial_E(\nu, F) = \lim_{n \to +\infty} \int_F f_{D_n} \, d\lambda \tag{9} \]
for every $F \in \Sigma$.

For any fixed $F \in \Sigma$ let $F_1 = F \cap E^c, F_2 = F \cap E$. We have that for every $D \in \Pi(F_1)$
\[ \int_{F_1} \left| \sum_{I \in D} \left[ \frac{\nu_E(I)}{\lambda(I)} - \frac{\partial^+_E(\nu, I)}{\lambda(I)} \right] \right| \, d\lambda = \sum_{I \in D} \int_I \left| \frac{\nu_E(I) - \partial^+_E(\nu, I)}{\lambda(I)} \right| \, d\lambda = \sum_{I \in D} \left| \nu_E(I) - \partial^+_E(\nu, I) \right| \tag{10} \]
asd analogously, for $D \in \Pi(F_2)$
\[ \int_{F_2} \left| \sum_{I \in D} \left[ \frac{\nu_E(I)}{\lambda(I)} - \frac{\partial^+_E(\nu, I)}{\lambda(I)} \right] \right| \, d\lambda = \sum_{I \in D} \left| \nu_E(I) - \partial^+_E(\nu, I) \right|. \tag{11} \]

Note now that
\[ f_{D_n} = \sum_{I \in D_n} \frac{\partial^+_E(\nu, I)}{\lambda(I)} - 1_I = \sum_{I \in D_n} \frac{\partial^+_E(\nu, I \cap E^c) + \partial^-_E(\nu, I \cap E)}{\lambda(I)} \]

Since $D_n = D'_n \cup D''_n$ where $D'_n \in \Pi(E^c)$ and $D''_n \in \Pi(E)$, we can also write
\[ f_{D_n} = \sum_{I \in D'_n} \frac{\partial^+_E(\nu, I)}{\lambda(I)} 1_I + \sum_{I \in D''_n} \frac{\partial^-_E(\nu, I)}{\lambda(I)} 1_I. \]

Thus
\[ \int_F |\psi^E_n - f_{D_n}| \, d\lambda = \sum_{I \in D_n} \int_I |\psi^E_n - f_{D_n}| \, d\lambda = \sum_{I \in D'_n} \int_I \left| \psi^E_n - \frac{\partial^+_E(\nu, I)}{\lambda(I)} \right| \, d\lambda + \sum_{I \in D''_n} \int_I \left| \psi^E_n - \frac{\partial^-_E(\nu, I)}{\lambda(I)} \right| \, d\lambda. \]

Then from (10) and (11) one derives
\[ \int_F |\psi^E_n - f_{D_n}| \, d\lambda = \sum_{I \in D'_n} \left| \nu_E(I) - \partial^+_E(\nu, I) \right| + \sum_{I \in D''_n} \left| \nu_E(I) - \partial^-_E(\nu, I) \right|. \]

Since $\delta(D'_n) \to 0$ and $\delta(D''_n) \to 0$,
\[ \lim_{n \to +\infty} \int_F |\psi^E_n - f_{D_n}| \, d\lambda = 0, \]
which, together with (9) shows that
\[ \partial_E(\nu, F) = \lim_{n \to +\infty} \int_F \psi^E_n \, d\lambda \]
for every $F \in \Sigma$. \qed

Note that since $\psi^E_n$ converges in $L^1(\lambda)$ and $\lambda$ is countably additive, a subsequence of it converges almost everywhere to $f$. Thus the above integral representation of $\partial_E(\nu, \cdot)$ immediately compares with Theorem 3 in [9].

The statement of Theorem 5.3 does not hold if $\nu$ is assumed to admit refinement differential, as the following example shows.
Example 5.2 Consider the space $\Omega$ and the game $\nu$ of Example 5.1 and take $E = \emptyset$, so we need to deal only with the outer differential; then we already know that $\nu$ admits refinement differential $\lambda$ at $\emptyset$, but has no BC differential.

Note that $\nu \ll 4 \lambda$ since $x \mapsto \sqrt{x}$ is absolutely continuous.

Now, if we choose $\left\{ [0, 1] \times \left[ 0, \frac{1}{2} \right], [0, 1] \times \left[ \frac{1}{2}, 1 \right] \right\}$ and keep halving the sets of $D_{n-1}$ by horizontal cuts, then $D_n = \left\{ [0, 1] \times \left[ \frac{r}{2^n}, \frac{r + 1}{2^n} \right], r = 0, \ldots, 2^n - 1 \right\}$ (apart for $r = 0$ for we take the first rectangle closed). Then according to the definition of $\nu$

\[ \psi_n^E = \varphi_n^E = \sum_{k=1}^{2^n} \frac{\nu(I_k)}{\lambda(I_k)} 1_{I_k} = \sum_{k=1}^{2^n} \frac{1}{\sqrt{\lambda(I_k)}} 1_{I_k} = \sum_{k=1}^{2^n} \sqrt{2^n} 1_{I_k} = \sqrt{2^n} \]

whence

\[ \int_F \psi_n^E d\lambda = \lambda(F) \sqrt{2^n} \to +\infty \]

for every $F \in \Sigma^+$. However, one immediately notes that if $D_1$ had been differently chosen, for instance

\[ D_1 = \left\{ \left[ 0, \frac{1}{2} \right] \times [0, 1], \left[ \frac{1}{2}, 1 \right] \times [0, 1] \right\}, \]

then the statement of the Theorem would have been fulfilled.

Hence a more difficult question would be: does it exist an example such that whichever sequence of partitions one takes, the corresponding sequence of projections does not converge in mean to the refinement differential?

So far we have not been able to find a suitable counterexample in this sense.

6 Calculus

Several results for differential calculus are scattered throughout the literature for both the refinement differential $\delta^E(\nu, \cdot)$ and for the $\mu$-derivative $\Delta^E(\nu, \cdot)$. We shall reobtain some of these rules for the BC differential, and we will compare them to the above two definitions.

We begin by proving a vector-valued version of the Chain Rule. In order to keep the notation as simple as possible we enounce it for the bidimensional case, anyway the same proof applies to the $n$-dimensional framework.

**Theorem 6.1** Let $\nu : \Sigma \to \mathbb{R}^2$ be a vector valued game, i.e. $\nu(\cdot) = (\nu'(\cdot), \nu''(\cdot))$, with $\nu', \nu'' : \Sigma \to \mathbb{R}$. Let $\lambda$ be as in Theorem 5.2, and both $\nu', \nu'' \ll 3 \lambda$ be BC differentiable at $E \in \Sigma$. Suppose moreover that $g : \mathbb{R}^2 \to \mathbb{R}$ is differentiable at $\nu(E)$.

Then $g \circ \nu : \Sigma \to \mathbb{R}$ admits BC differential at $E$ and

\[ \partial^E(g \circ \nu, \cdot) = \nabla g(\nu(E)) \cdot \partial^E(\nu, \cdot) \]

where $\partial^E(\nu, \cdot) = (\partial^E(\nu', \cdot), \partial^E(\nu'', \cdot))$. 

22
Proof. As usual we work with outer differentials, and for vectors in \( \mathbb{R}^2 \) we use the maximum norm. Fix \( F \subset E^c \). For a decomposition \( D \in \Pi(F) \) we estimate:

\[
\sum_{I \in D} |(g \circ \nu)_E(I) - \nabla g(\nu(E)) \cdot \partial E^+(\nu, I)|.
\]

First of all note that if \( \|\nu_E(I)\| = 0 \), then \( (g \circ \nu)_E(I) = 0 \) (in fact \( (g \circ \nu)_E(I) = g[\nu_E(I) + \nu(E)] - g[\nu(E)] \)). Hence we can assume without loss of generality \( \|\nu_E(I)\| \neq 0 \) for every \( I \in D \). Write now the previous expression as:

\[
\sum_{I \in D} |(g \circ \nu)_E(I) - \nabla g(\nu(E)) \cdot \nu_E(I) + \nabla g(\nu(E)) \cdot \nu_E(I) - \nabla g(\nu(E)) \cdot \partial E^+(\nu, I)| \leq \\
\sum_{I \in D} |\nabla g(\nu(E)) \cdot (\nu_E(I) - \partial E^+(\nu, I))| + \sum_{I \in D} |(g \circ \nu)_E(I) - \nabla g(\nu(E)) \cdot \nu_E(I)| \leq \\
\|\nabla g(\nu(E))\| \sum_{I \in D} \|\nu_E(I) - \partial E^+(\nu, I)\| + \sum_{I \in D} \|\nu_E(I)\| \frac{|(g \circ \nu)_E(I) - \nabla g(\nu(E)) \cdot \nu_E(I)|}{\|\nu_E(I)\|}.
\]

The BC differentiability of \( \nu' \) and \( \nu'' \) implies that \( \sum_{I \in D} \|\nu_E(I) - \partial E^+(\nu, I)\| \) can be made suitably small.

Furthermore, as \( \nu', \nu'' \ll_3 \lambda \) implies \( \nu'_E, \nu''_E \ll_2 \lambda \), for suitably small \( \lambda(I) \) the increment \( \|\nu_E(I)\| \) can be made suitably small too and hence, by the differentiability of \( g \) at \( \nu(E) \) we obtain

\[
\frac{|(g \circ \nu)_E(I) - \nabla g(\nu(E)) \cdot \nu_E(I)|}{\|\nu_E(I)\|} < \varepsilon.
\]

Moreover we have:

\[
\|\nu_E(I)\| \leq \|\nu_E(I) - \partial E^+(\nu, I)\| + \|\partial E^+(\nu, I)\|
\]

and hence, choosing \( \delta(D) < \delta(\varepsilon) = \min\{\delta'(\varepsilon), \delta''(\varepsilon)\} \), where \( \delta', \delta'' \) are the the parameters of BC differentiability of \( \nu' \) and \( \nu'' \) respectively, we get

\[
\sum_{I \in D} |(g \circ \nu)_E(I) - \nabla g(\nu(E)) \cdot \partial E^+(\nu, I)| \leq \\
\varepsilon \|\nabla g(\nu(E))\| + \varepsilon \sum_{I \in D} \|\nu_E(I)\| \leq \\
\varepsilon \|\nabla g(\nu(E))\| + \varepsilon \sum_{I \in D} \|\nu_E(I) - \partial E^+(\nu, I)\| + \sum_{I \in D} \|\partial E^+(\nu, I)\| \leq \\
\varepsilon \|\nabla g(\nu(E))\| + \varepsilon^2 + \varepsilon \sum_{I \in D} \|\partial E^+(\nu, I)\| \leq \\
\varepsilon \|\nabla g(\nu(E))\| + \varepsilon^2 + \varepsilon [\text{var} \partial E^+(\nu', :)(\Omega) + \text{var} \partial E^+(\nu'', :)(\Omega)].
\]

This concludes the proof.

In the previous result the \( \ll_3 \) hypothesis is needed for both the components of \( \nu \), as shown by the following
Example 6.1 Consider the pair of finitely additive measures in Example 3.7 of [7]; these are two finitely additive measures, say \( \nu', \lambda \) where \( \lambda \) is the usual Lebesgue measure on \([0,1]\) and \( \nu' \) is a finitely additive measure such that \( \nu' \perp \lambda \); more precisely the range of the pair \((\lambda, \nu')\) is

\[
\{(0,1] \times [0,1] \cup ([0,1] \times \{0\}) \cup \{0,1\} \times \{1\}\}.
\]

Then \( \nu' \) admits BC differential at each \( E \in \Sigma \), for it is additive, and \( \partial E(\nu', \cdot) = \nu' \).

However the game \([\nu']^2\) has no BC differential at \( \varnothing \), since there are partitions \( D \in \Pi(\Omega) \) such that \( \lambda(I) \) is arbitrarily small and \( \nu'(I) = 1 \); since for these sets

\[
||\nu'\|_0^2(I) - 2 \nu'(\varnothing) \partial_{\varnothing}^+(\nu', I)\| = ||\nu'\|_0^2(I) = 1
\]

the corresponding sums \( \sum_{I \in D} ||\nu'\|_0^2(I) - 2 \nu'(\varnothing) \partial_{\varnothing}^+(\nu', I)\| \neq 0 \). In fact \( \nu' \ll_1 \lambda \) but \( \nu' \not\ll_2 \lambda \) which in the finitely additive case precisely coincides with the \( \ll_4 \) absolute continuity.

Now take \( E = \varnothing \) and \( g(x,y) = x^2 \), and take \( \nu = (\nu', \nu'') \), with \( \nu'' \ll_3 \lambda \) whatever and BC differentiable. Then \((g \circ \nu) = [\nu']^2\) does not admit BC differential at \( \varnothing \) while \( \nu' \) and \( \nu'' \) do.

Now, all differentiation rules for BC differential follow easily.

**Corollary 6.1** Let \( \lambda \) be as in Theorem 5.2, and let \( \nu' \) and \( \nu'' \) admit BC differentials at some \( E \in \Sigma \), with \( \nu', \nu'' \ll_3 \lambda \).

i. If \( \alpha, \beta \) are any two numbers in \( \mathbb{R} \), the game \( \alpha \nu' + \beta \nu'' \) admits BC differential at \( E \) and

\[
\partial E(\alpha \nu' + \beta \nu'', \cdot) = \alpha \partial E(\nu', \cdot) + \beta \partial E(\nu'', \cdot).
\]

ii. The product \( \nu' \nu'' \) has BC differential at \( E \) and

\[
\partial E(\nu' \nu'', \cdot) = \nu'(E) \partial E(\nu'', \cdot) + \nu''(E) \partial E(\nu', \cdot).
\]

iii If \( \nu'(E) \neq 0 \) then the game

\[
\mu(F) = \begin{cases} 
1 & \text{if } \nu'(F) \neq 0 \\
0 & \text{if } \nu'(F) = 0
\end{cases}
\]

admits BC differential at \( E \), and

\[
\partial E(\mu, \cdot) = -\frac{\partial E(\nu', \cdot)}{||\nu'(E)||^2}.
\]

iv. If \( \nu'(E) \neq 0 \) then the game

\[
\mu'(F) = \begin{cases} 
\frac{\nu''(F)}{\nu'(F)} & \text{if } \nu'(F) \neq 0 \\
0 & \text{if } \nu'(F) = 0
\end{cases}
\]
admits BC differential at $E$, and
\[
\partial_E(\mu', \cdot) = \frac{\nu'(E)\partial_E(\nu'', \cdot) - \nu''(E)\partial_E(\nu', \cdot)}{[\nu'(E)]^2}.
\] (16)

**Proof.** i. Take $\nu(\cdot) = (\nu'(\cdot), \nu''(\cdot))$ and let $g(x, y) = \alpha x + \beta y$. Then $(g \circ \nu)(\cdot) = \alpha \nu'(\cdot) + \beta \nu''(\cdot)$. Hence $g \circ \nu$ is BC differentiable at $E$ and, by Theorem 6.1, i. follows immediately. The product rule ii. can be proven analogously.

iii. Apply Theorem 6.1 with
\[
g(x, y) = \begin{cases} 
\frac{1}{x} & \text{if } x \neq 0 \\
0 & \text{if } x = 0
\end{cases}
\]

iv. Apply Theorem 6.1 with
\[
g(x, y) = \begin{cases} 
\frac{y}{x} & \text{if } x \neq 0 \\
0 & \text{if } x = 0
\end{cases}
\]

and use ii. and iii.

**Remark 6.1** The differentiation rules can also be obtained in a direct way, and the scalar chain rule proven independently. This approach, in spite of the fact that proofs become considerably longer, has lighter assumptions. Indeed, to prove i. one need not to assume any form of absolute continuity, while for the remaining rules to hold, one just has to ask $\nu' \ll_3 \lambda$.

Here we have provided the shorter, though less general proof, for the sake of brevity.

It is interesting to observe that, as for ii. proved with lighter assumptions, these can not be further reduced: in fact ii. does not hold without asking $\nu' \ll_3 \lambda$ (which is conversely not needed for the analogous relationship for the refinement differentials): it is enough to take $\nu', \lambda$ as in Example 6.1 and $g(x) = x^2$. Then $(g \circ \nu')^2$ does not admit BC differential at $\emptyset$, while $\nu'$ does.

Also worth mentioning is the fact that the $\mu$-differentiability of Definition 5.3 above is not preserved by every operation; it is quite immediate to get convinced that any linear combination of $\mu$-differentiable games is also $\mu$-differentiable, and clearly the same Calculus Rule holds. However the following Example shows that the product of $\mu$-differentiable games may fail to be $\mu$-differentiable, even when the conditions of statement ii. of Corollary 6.1 (in the more general form explained in the Remark) are fulfilled.

**Example 6.2** Let $\Omega, \Sigma, \lambda$ and $\nu$ be as in the previous Example 6.1.

Let $\nu' = \lambda^2, \nu'' = \nu, \mu = \lambda$. Then $\nu' \ll_3 \lambda$, and for every $E, F, G \in \Sigma$ with $G \subseteq E, F \subseteq E^c$ and $\lambda(F \cup G) > 0$ consider
\[
\frac{|\nu'(E \cup F \setminus G) - \nu'(E) - 2\lambda(E)\lambda(F) + 2\lambda(E)\lambda(G)|}{\lambda(F \cup G)}.
\]
Suppose that $\nu$ is a monotone game, with $\nu \ll \lambda$ and let $A, B \in \Sigma$ with $A \subseteq B$; let
$$\Sigma_{A,B} = \{ C \in \Sigma | A \subseteq C \subseteq B \}.$$ 
Suppose that

- $\nu$ is refinement differentiable at each $I \in \Sigma_{A,B}$;
- $\nu(B) - \nu(A) \leq \nu(A)$

Then for every $\varepsilon > 0$ there exist $C, E \in \Sigma_{A,B}$ such that
$$\delta_{E}(\nu, E \setminus A) - \varepsilon \lambda(B \setminus A) \leq \nu(B) - \nu(A) \leq \delta_{C}(\nu, C \setminus A) + \varepsilon \lambda(B \setminus A).$$  \hspace{1cm} (17)
Proof. Without loss of generality we can assume that $\nu(\Omega) \leq 1$.
If $\nu(A) = 0$ then immediately $\nu(B) = 0$ too, and one can choose $C = E = A$ to achieve the conclusion.

The same argument applies if $\nu(B) - \nu(A) = 0$.
Therefore we assume that $\nu(A) > 0$ and $\nu(B) - \nu(A) > 0$ too. Then also $\lambda(B \setminus A) > 0$.
Without loss of generality, we can also assume that $\varepsilon < \frac{1}{2\nu(\Omega)\lambda(\Omega)}$.
Thus, since $\nu(X) \lambda(X) \leq \nu(\Omega)\lambda(\Omega)$, this implies that $\varepsilon < \frac{1}{2\lambda(X)\nu(X)}$ whenever $\nu(X) \lambda(X) \neq 0$.
Since $\nu$ is monotone, $\nu(X) > 0$ for each $X \in \Sigma_{A,B}$; hence we can define $\sigma : \Sigma_{A,B} \to \mathbb{R}$ as
$$\sigma(X) = \frac{\lambda(X \setminus A)}{\nu(X)} - 2\varepsilon \lambda(B \setminus A) \lambda(X).$$

By assumptions, $\sigma$ is continuous with respect to the pseudo-metric $\rho(E,F) = \varepsilon \lambda(B \setminus A) \lambda(E \Delta F)$.
Observe that $\sigma(A) < 0$, whence
$$a = \inf\{\sigma(X), X \in \Sigma_{A,B}\} < 0$$
while $\sigma(B) = \lambda(B \setminus A) \left[\frac{1}{\nu(B)} - 2\varepsilon \lambda(B)\right] > 0$.

Take $\eta > \sigma(B) - a > 0$. Then $\sigma(B) < \eta + a$; let $\xi < \eta \land \frac{\rho(A,B)}{2}$.
Apply the Ekeland Principle to the map $\sigma$ on $(\Sigma_{A,B}, \rho)$ with $x_1 = B$ (see [14] page 172). Then there exists $C \in \Sigma_{A,B}$ such that $\rho(C,B) \leq \xi$ and
$$\sigma(X) > \sigma(C) - \frac{\eta}{\xi} \rho(C,X) \quad (18)$$
for each $X \in \Sigma_{A,B}$.
As $\rho(C,B) \leq \xi < \frac{\rho(A,B)}{2}$ necessarily $\lambda(B \setminus C) < \frac{\lambda(B \setminus A)}{2}$; hence $A \not\subset C$ and $C \not\subset B$.
(18) can be written as
$$\sigma(C) < \sigma(X) + \frac{\eta}{\xi} \varepsilon \lambda(B \setminus A) \lambda(X \Delta C)$$
for every $X \in \Sigma_{A,B}$. Then the first order condition for
$$\varphi(X) = \sigma(X) + \frac{\eta}{\xi} \varepsilon \lambda(B \setminus A) \lambda(X \Delta C)$$
is $\delta_C^-(\varphi, C \setminus A) \leq 0$. By the differentiation rules one computes
$$\delta_C^-(\varphi, I) = \delta_C^-(\sigma, I) + \frac{\eta}{\xi} \varepsilon \lambda(B \setminus A) \lambda(I) = \frac{\lambda(I) \nu(C) - \lambda(C \setminus A) \delta_C^-(\nu, I)}{\nu^2(C)} - \left(2 - \frac{\eta}{\xi}\right) \varepsilon \lambda(B \setminus A) \lambda(I).$$

Note that the choice of $\xi < \eta$ implies that $\alpha := 2 - \frac{\eta}{\xi} < 1$. Hence, from $\delta_C^-(\varphi, C \setminus A) \leq 0$ we have
$$\frac{\lambda(C \setminus A) \nu(C) - \lambda(C \setminus A) \delta_C^-(\nu, C \setminus A)}{\nu^2(C)} - \alpha \varepsilon \lambda(B \setminus A) \lambda(C \setminus A) \leq 0$$
that is
\[ \lambda(C \setminus A) \left[ \frac{\nu(C) - \delta_C^-(\nu, C \setminus A)}{\nu^2(C)} - \alpha \varepsilon \lambda(B \setminus A) \right] \leq 0. \]

Observe that \( \lambda(C \setminus A) > 0 \), for otherwise one would reach a contradiction, since
\[ \rho(A, B) \leq \rho(A, C) + \rho(C, B) \leq \xi \leq \frac{\rho(A, B)}{2}. \]

Hence necessarily
\[ \frac{\nu(C) - \delta_C^-(\nu, C \setminus A)}{\nu^2(C)} - \alpha \varepsilon \lambda(B \setminus A) \leq 0. \]

Now, by our assumptions, \( \nu(C) \geq \nu(B) - \nu(A) \) and \( \nu^2(C) > 0 \); thus
\[ \frac{\nu(B) - \nu(A) - \delta_C^-(\nu, C \setminus A)}{\nu^2(C)} - \alpha \varepsilon \lambda(B \setminus A) \leq 0. \]

As \( \alpha < 1 \), a fortiori one finds
\[ \frac{\nu(B) - \nu(A) - \delta_C^-(\nu, C \setminus A)}{\nu^2(C)} - \varepsilon \lambda(B \setminus A) \leq 0. \]  \hspace{1cm} (19)

Now, if \( \nu(B) - \nu(A) - \delta_C^-(\nu, C \setminus A) \leq 0 \), this means that
\[ \nu(B) - \nu(A) - \delta_C^-(\nu, C \setminus A) \leq \varepsilon \lambda(B \setminus A) \]

so the right hand side of (17) is satisfied. If \( \nu(B) - \nu(A) - \delta_C^-(\nu, C \setminus A) > 0 \) having assumed that \( \nu(\Omega) \leq 1 \), also \( \nu^2(C) \leq 1 \); then from (19)
\[ [\nu(B) - \nu(A)] - \delta_C^-(\nu, C \setminus A) \leq \frac{[\nu(B) - \nu(A)] - \delta_C^-(\nu, C \setminus A)}{\nu^2(C)} \leq \varepsilon \lambda(B \setminus A) \]

which again proves the right hand side of (17).

As announced, we are now able to prove a form of the Fundamental Theorem of Calculus for the refinement differential, analogous to that given in [12] for the \( \mu \)-differential.

**Theorem 6.2** Let \( \lambda \) be a strongly non atomic measure, \( \nu \) a monotone game, with \( \nu \ll_3 \lambda \) and let \( A, B \in \Sigma \) with \( A \subseteq B \). Suppose that \( \nu \) is refinement-differentiable at each \( E \in \Sigma_{A,B} \). Then for every \( \varepsilon > 0 \) there exists \( D \in \Pi(B \setminus A) \) say \( D = \{ I_1, \ldots, I_n \} \) and two chains in \( \Sigma_{A,B} \), \( A \subseteq C_1 \subseteq \ldots \subseteq C_n \subseteq B \), and \( A \subseteq E_1 \subseteq \ldots \subseteq E_n \subseteq B \), with \( F_k \subseteq C_k \cup E_k \subseteq F_k \), where
\[ F_k = \bigcup_{\ell=0}^k I_\ell, k = 1, \ldots, n \ (I_0 = A) \] such that
\[ \sum_k \delta_{E_k}^-(\nu, F_k \setminus E_k) - \varepsilon \lambda(B \setminus A) \leq \nu(B) - \nu(A) \leq \sum_k \delta_{C_k}^-(\nu, F_k \setminus C_k) + \varepsilon \lambda(B \setminus A). \]  \hspace{1cm} (20)

**Proof.** We shall divide our proof into two cases.

**First case:** \( \nu(A) > 0 \).

28
If \( \nu(B) - \nu(A) \leq \nu(A) \) this is simply a consequence of Lemma 6.1 above. Assume then that 
\[ \nu(B) - \nu(A) > \nu(A). \]
Clearly \( \lambda(A) < \lambda(B) \) (otherwise, if \( \lambda(B \setminus A) = 0 \), by \( \nu \ll \lambda \) also \( \nu(B) = \nu(A); \) contradiction).

Let \( \tau = \tau(\varepsilon) \) be determined by the absolute continuity \( \nu \ll \lambda \).

Choose any \( D \in \Pi(B \setminus A) \) with mesh \( \delta(D) \leq \tau[\nu(A)] \), say \( D = \{I_1, \ldots, I_n\}, I_o = A. \)

Then, since \( I_k = F_{k+1} \setminus F_k \) we have that 
\[
\nu(F_{k+1}) - \nu(F_k) \leq \nu(A) \leq \nu(F_k)
\]
for \( k = 0, \ldots, n - 1. \) Therefore we can apply Lemma 6.1 to each increment \( \nu(F_{k+1}) - \nu(F_k). \)

Writing
\[
\nu(B) - \nu(A) = \sum_{k=1}^{n} [\nu(F_k) - \nu(F_{k-1})]
\]
we reach the desired relationship (20).

**Second case:** \( \nu(A) = 0. \)

If \( \nu(B) = 0 \) too, we can set again \( D = \{B \setminus A\}, E_1 = C_1 = B \setminus A \) and again (20) is immediate.

Assume then that \( \nu(B) > 0. \) Note that \( \lambda(B \setminus A) > 0 \) (for otherwise, by the absolute continuity, 
\( \nu(B) - \nu(A) = 0, \) and since we are assuming \( \nu(A) = 0, \) this in turn would imply the contradiction
\( \nu(B) = 0). \)

Without loss of generality, we can choose \( \varepsilon > 0 \) so that \( \frac{\varepsilon}{2} \lambda(B \setminus A) < \nu(B). \)

First we shall prove that for every \( t \in [0, \nu(B)] \) there exists a set \( E \in \Sigma_{A,B} \) such that \( \nu(E) = t. \)

Indeed, we can consider a filtering family \( \{(B \setminus A)_t \}, t \in [0, 1]\) according to Lemma 2.1 in [7],

such that \( (B \setminus A)_o = \emptyset, (B \setminus A)_1 = B \setminus A; t < t' \) implies that \( (B \setminus A)_t \subset (B \setminus A)_{t'}; \) and finally
\( \lambda((B \setminus A)_t) = t \lambda(B \setminus A). \)

Then the map \( \gamma(t) = \nu[A \cup (B \setminus A)_t] \) is continuous on \([0, 1], \) for
\[
|\gamma(t) - \gamma(t')| = |\nu[A \cup (B \setminus A)_t] - \nu[A \cup (B \setminus A)_{t'}]|
\]
with \( \lambda\{[A \cup (B \setminus A)_t] \} - \Delta [A \cup (B \setminus A)_{t'}] = |t - t'| \lambda(B \setminus A). \) Moreover \( \gamma(0) = 0, \gamma(1) = \nu(B). \)

Choose then \( t_o \) such that \( \nu[A \cup (B \setminus A)_{t_o}] = \frac{\varepsilon}{2} \lambda(B \setminus A). \) Then we can apply our first case to the 
sets \( B, A_1 = A \cup (B \setminus A)_{t_o} \) since \( \nu(A_1) > 0, \) and determine \( D \in \Pi(B \setminus A_1) \) corresponding to \( \frac{\varepsilon}{2}. \)

On the other side
\[
\nu(A_1) - \nu(A) = \nu(A_1) = \frac{\varepsilon}{2} \lambda(B \setminus A) = \frac{\varepsilon}{2} \lambda(B \setminus A) + \delta_{A_1}(\nu, A_1 \setminus A_1)
\]
and \( \nu(A_1) > \delta_{A_1}(\nu,A_1 \setminus A_1) - \frac{\varepsilon}{2} \lambda(B \setminus A). \)

Summing up we reach
\[
\nu(B) - \nu(A) = \nu(B) - \nu(A_1) + \nu(A_1) \leq \sum_k \delta_{C_k}(\nu,F_k \setminus C_k) + \frac{\varepsilon}{2} \lambda(B \setminus A_1) + \frac{\varepsilon}{2} \lambda(B \setminus A) + \delta_{A_1}(\nu,A_1 \setminus A_1) \leq
\]
29
\[
\leq \sum_k \delta_{C_k}^-(\nu, F_k \setminus C_k') + \varepsilon\lambda(B \setminus A)
\]
and analogously
\[
\nu(B) \geq \sum_k \delta_{E_k}^- (\nu(F_k \setminus E_k') - \varepsilon\lambda(B \setminus A)).
\]
This concludes the proof. \qed

\section{The core of Burkill integrable games}

In this section we shall consider the space $\mathbb{S}$ of games that are Burkill-Cesari integrable and we shall investigate their core.

We begin with a Lemma

\textbf{Lemma 7.1} If $\nu \in \mathbb{S}$ has non empty core, then for every $m \in \text{core}(\nu)$ one has $\int \nu \leq m$ setwise on $\Sigma$.

The proof is the same as that of Lemma 4.1 of [13].

This immediately yields the following result.

\textbf{Proposition 7.1} If $\nu \in \mathbb{S}$ has feasible integral, that is $\int_\Omega \nu = \nu(\Omega)$, and $\text{core}(\nu)$ is non-empty, then $\partial_\emptyset(\nu, \cdot) = \int \nu$ and $\text{core}(\nu) = \{ \partial_\emptyset(\nu, \cdot) \}$.

\textbf{Proof.} Observe first that by \textbf{Proposition 5.1}, it immediately follows that $\eta = \int \nu$ is a non-negative finitely additive measure.

By the previous Lemma, if $m \in \text{core}(\nu)$, the pair of finitely additive measures $\eta$ and $m$ fulfills $\eta \leq m$; thus if $\eta(\Omega) = \nu(\Omega) = m(\Omega)$ necessarily $\eta = m$.

Thus $\text{core}(\nu) = \{ \eta \}$. Finally, as already noted $\eta = \partial_\emptyset(\nu, \cdot)$. \qed

We shall now introduce a new concept.

\textbf{Definition 7.1} A game $\nu$ on $(\Omega, \Sigma)$ is said to be \textit{pseudoadditive} if for every $\varepsilon > 0$ there exists a partition $D_\varepsilon \in \Pi(\Omega)$ with $\delta(D_\varepsilon) < \varepsilon$ and

\[
\left| \nu(\Omega) - \sum_{J \in D_\varepsilon} \nu(J) \right| < \varepsilon.
\]

The following statement justifies the introduction of such a concept.

\textbf{Proposition 7.2} If $\nu \in \mathbb{S}$ is pseudoadditive, then $\int \nu$ is feasible.
Proof. Fix $\varepsilon > 0$ and, by means of the Burkill-Cesari integrability, choose $\delta(\varepsilon) < \varepsilon$ such that for each $D \in \Pi(\Omega)$ with $\delta(D) < \delta$ there holds

$$\left| \sum_{J \in D} \nu(J) - \eta(\Omega) \right| < \varepsilon.$$ 

Among these decompositions, pick one such that

$$\left| \nu(\Omega) - \sum_{J \in D} \nu(J) \right| < \delta < \varepsilon.$$ 

Then immediately $|\nu(\Omega) - \eta(\Omega)| < 2\varepsilon$. \qed

A pseudoadditive game may fail to be Burkill-Cesari integrable, as the following example shows.

**Example 7.1** In [7], (Osservazione 1.8) the following example is given. On $I = [0,1]$ let $m : 2^I \to [0,1]$ be any finitely additive extension of the Lebesgue measure, and let $\varphi$ be the measure game $\varphi = \sqrt{m}$. Let $\Omega = [0,1]$ and consider for every $E \subset \Omega$ and every $y \in [0,1]$ the section $E_y = \{x \in I : (x,y) \in E\}$. Then as shown in [7] the game on $2^\Omega$

$$\nu(E) = \int_0^1 \varphi(E_y)dm(y)$$

is subadditive and nonadditive; furthermore $\nu$ is *semiconvex*, namely for every $E \subset \Omega$ there is $F \subset E$ such that $\nu(F) = \nu(E \setminus F) = \frac{1}{2} \nu(E)$.

Note that if $E$ is a rectangle $E = [a,b] \times [c,d]$ then $\nu(E) = (d-c)\sqrt{b-a}$, and the same holds if the rectangle is open, or it only contains one vertical (resp. horizontal) side; thus if $\mu$ denotes any finitely additive non atomic extension of the usual Lebesgue measure on the power set $2^\Omega$ on rectangles one finds $\mu(R) \leq \nu(R)$.

Let us prove that $\nu$ is pseudoadditive with respect to the mesh $\delta_\mu$.

Fix $\varepsilon > 0$ and choose $n \in \mathbb{N}$ such that $\frac{1}{2^n} < \varepsilon$; consider the $2^{n+1}$ points of $[0,1]$ defined as $x_i = \frac{i}{2^n}, i = 0, \ldots, 2^n$, and take the corresponding $2^n$ strips $R_i = [0,1] \times [x_i, x_{i+1}], i = 0, \ldots, 2^n - 1$ (in the case of $i = 2^n - 1$ take the closed interval $[x_i, x_{i+1}]$ instead).

The decomposition $D = \{R_i, i = 0, \ldots, 2^n - 1\}$ then is such that $\delta_\mu(D) < \varepsilon$ for each $R_i$ is a rectangle of $\frac{1}{2^n} \mu$-measure, and according to the above computation $\nu(R_i) = \frac{1}{2^n}$. Therefore

$$\sum_i \nu(R_i) = 2^n \nu(R_1) = 1 = \nu(\Omega).$$

Because of semiconvexity, it is easy to get convinced that $\nu$ is not Burkill-Cesari integrable; indeed by the above construction, one can prove that $\mu \leq \nu$ at least on the Borel $\sigma$-algebra $\mathcal{B}_\Omega$; on the other side, by semiconvexity, each set $E \subset \Omega$ can be always decomposed into $2^n$ pairwise disjoint subsets, each of measure $\frac{\nu(E)}{2^n}$. Since, these sets are obtained by iterative “halving” of the whole
E, and, as found in [7], the halving of a set E is of the form \( F = \bigcup_{y \in [0,h]} A_y \times \{y\} \) it can be shown that if E is in the Borel \( \sigma \) algebra, the above decomposition is also done of borelian sets. Therefore such a decomposition \( D \) has \( \delta_{\mu}(D) \leq \frac{\nu(E)}{2^n} \) and \( \sum_{E_i \in D} \nu(E_i) = 2^n \nu(E_1) = \nu(E) \).

If \( \nu \) were Burkill-Cesari integrable, then for \( n \) suitably large \( \sum_{E_i \in D} \nu(E_i) - \eta(E) < \epsilon \) and hence eventually \( \nu(E) = \eta(E) \); this in turn would imply that \( \nu \) is additive on \( B_\Omega \) which is false, as it can be immediately checked.

Pseudoadditive games include a class of well-known and widely used games

**Definition 7.2** Let \( P : \Sigma \to \mathbb{R}^n_+ \) be a semiconvex finitely additive measure, and let \( g : R(P) \to \mathbb{R} \) be positively homogeneous of degree 1. Then the game \( \nu = g \circ P \) is said to be a quasi-market game. If furthermore \( \nu \) is superadditive, then we refer to it as a market game.

The above class is actually slightly larger that of classical market games, for we have assumed \( P \) to be only finitely additive. Remind that in this case semiconvexity is equivalent to strong non nonatomicity.

**Proposition 7.3** If \( \lambda \) is finitely additive and \( \nu = g \circ P \) is a quasi-market game, then \( \nu \) is pseudoadditive.

**Proof.** The vector finitely additive measure \( (P, \lambda) \) has a filtering family \( \{\Omega_t, t \in [0,1]\} \) such that \( (P, \lambda)(\Omega_t) = t(P, \lambda)(\Omega) \) (see [8], Lemma 2.2).

Then for fixed \( \epsilon > 0 \) let \( n_0 \in \mathbb{N} \) be such that \( \frac{\lambda(\Omega)}{n_0} < \epsilon \), and divide \( \Omega \) into \( n_0 \) pairwise disjoint sets \( \Omega_i \), each with \( (P, \lambda)(\Omega_i) = \frac{(P, \lambda)(\Omega)}{n_0} \).

Thus for \( D = \{\Omega_1, \ldots, \Omega_{n_0}\} \) one has \( \delta(D) < \epsilon \) and

\[
\left| \nu(\Omega) - \sum_{i=1}^{n_0} \nu(\Omega_i) \right| = \left| g[P(\Omega)] - \sum_{i=1}^{n_0} g[P(\Omega_i)] \right| = \left| g[P(\Omega)] - n_0 g \left[ \frac{P(\Omega)}{n_0} \right] \right| = 0.
\]

\( \square \)

**References**


