The mesh role in preconditioning Finite Element matrix sequences

Cristina Tablino-Possio

joint work with Alessandro Russo

Dipartimento di Matematica e Applicazioni,
Università di Milano-Bicocca, Milano, Italy

Giornate di Algebra Lineare Numerica e Applicazioni
Perugia, 16-17 febbraio 2009

Outline

The Problem - Variational Form

Convection-Diffusion Equations

$$\begin{cases} \operatorname{div}\left(-a(\mathbf{x})\nabla u + \vec{\beta}(\mathbf{x})u\right) = f, \quad \mathbf{x} \in \Omega, \\ u_{|\partial\Omega} = 0. \end{cases}$$

Variational form

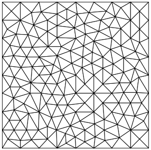
$$\begin{cases} \text{find } u \in H_0^1(\Omega) \text{ such that} \\ \int_\Omega \left(a\nabla u \cdot \nabla\varphi - \vec{\beta} \cdot \nabla\varphi\ u\right) = \int_\Omega f\varphi \quad \text{for all } \varphi \in H_0^1(\Omega). \end{cases}$$

Regularity Assumptions

$$\begin{cases} a \in \mathbf{C}^2(\overline{\Omega}), & \text{with } a(\mathbf{x}) \geq a_0 > 0, \\ \vec{\beta} \in \mathbf{C}^1(\overline{\Omega}), & \text{with } \operatorname{div}(\vec{\beta}) \geq 0 \text{ pointwise in } \Omega, \\ f \in L^2(\Omega). \end{cases}$$

The Problem - FE Approximation - I

Let
$\mathcal{T}_h = \{K\}$ finite element partition of $\overline{\Omega}$, polygonal domain, into triangles,
$h_K = \mathrm{diam}(K)$,
$h = \max_K h_K$.



We consider the space of linear finite elements
$V_h = \{\varphi_h : \overline{\Omega} \to \mathbb{R}$ s.t. φ_h is continuous, $\varphi_{h|_K}$ is linear, and $\varphi_{h|\partial\Omega} = 0\} \subset H^1_0(\Omega)$

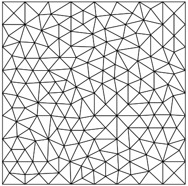
with basis

$$\varphi_i \in V_h \text{ s.t. } \varphi_i(\text{node } j) = \delta_{i,j},\; i,j = 1,\ldots,n(h),$$
$$n(h) = \dim(V_h) = \text{ number of the internal nodes of } \mathcal{T}_h.$$

The Problem - FE Approximation - II

The variational equation becomes

$$A_n(a, \vec{\beta})\mathbf{u} = \mathbf{b}$$



$\mathcal{T}_h = \{K\}$

with

$$A_n(a,\vec{\beta}) = \sum_{K\in\mathcal{T}_h} A_n^K(a,\vec{\beta}) \quad = \quad \Theta_n(a) + \Psi_n(\vec{\beta}) \in \mathbb{R}^{n\times n}, \quad n = n(h),$$

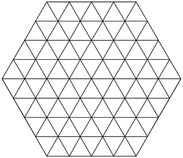
$$(\Theta_n(a))_{i,j} \quad = \quad \sum_{K\in\mathcal{T}_h} \int_K a \,\nabla\varphi_j \cdot \nabla\varphi_i \qquad \text{diffusive term},$$

$$(\Psi_n(\vec{\beta}))_{i,j} \quad = \quad -\sum_{K\in\mathcal{T}_h} \int_K (\vec{\beta}\cdot\nabla\varphi_i)\,\varphi_j \qquad \text{convective term},$$

and with suitable quadrature formulas in the case of non constant a and $\vec{\beta}$.

Motivations

Motivations



A good mesh generator will locally produce a partitioning, which is "asymptotically" similar to a partitioning into equilateral triangles, away from the boundary.

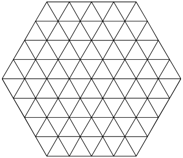
Aim:

To study the effectiveness of the proposed preconditioning strategy applied to FE approximations of Convection-Diffusion Eqns. both from the theoretical and numerical point of view.

We prove the PCG optimality and we give additional results about PGMRES convergence.

Motivations

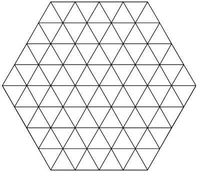
Motivations

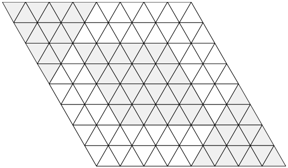


A good mesh generator will locally produce a partitioning, which is "asymptotically" similar to a partitioning into equilateral triangles, away from the boundary.

Aim:
To study the effectiveness of the proposed preconditioning strategy applied to FE approximations of Convection-Diffusion Eqns. both from the theoretical and numerical point of view.
We prove the PCG optimality and we give additional results about PGMRES convergence.

Mesh role - Equilateral elements - I





Due to the very special choice of the domain Ω, we have

$$\Theta_n(1) = \Pi \, T_N(\tilde{f}) \, \Pi^T$$

with
$T_N(\tilde{f})$ Toeplitz matrix related to a parallelogram shaped domain $\Omega_N \supseteq \Omega$ generated by

$$\tilde{f}(s_1, s_2) = \sqrt{3}(6 - 2\cos(s_1) - 2\cos(s_2) - 2\cos(s_1 + s_2))/3, \, (s_1, s_2) \in \mathcal{D} = (-\pi, \pi]^2,$$

and $\Pi \in \mathbb{R}^{n \times N}$, $n \leq N$ projection matrix.

By considering a smaller parallelogram shaped domain $\Omega_{\tilde{N}} \subseteq \Omega$, we have also

$$T_{\tilde{N}}(\tilde{f}) = \tilde{\Pi} \, \Theta_n(1) \, \tilde{\Pi}^T.$$

Thus

$$\lambda_{\min}(T_N(\tilde{f})) \leq \lambda_{\min}(\Theta_n(1)) \leq \lambda_{\min}(T_{\tilde{N}}(\tilde{f})).$$

Mesh role - Equilateral elements - II

Now, let $f(s_1, s_2) = 4 - 2\cos(s_1) - 2\cos(s_2)$ be the Toeplitz generating function in the case of FE approximations on a square $\Omega = (0,1)^2$ with Friedrichs-Keller meshes, or standard FD discretizations.

Since these two functions are equivalent, i.e.,

$$\frac{\sqrt{3}}{3} f \leq \tilde{f} \leq \sqrt{3} f \quad \text{on } \mathcal{D} = (-\pi, \pi]^2,$$

by virtue of LPO properties, the corresponding Toeplitz matrix sequences are spectrally equivalent, i.e.,

$$\frac{\sqrt{3}}{3} T_n(f) \leq T_n(\tilde{f}) \leq \sqrt{3}\, T_n(f) \quad \text{for any } n.$$

Remark: $\tilde{f}$ is the most natural function from the FE point of view, since no contribution is lost owing to the gradient orthogonality.

Mesh role - Equilateral elements - II

Now, let $f(s_1, s_2) = 4 - 2\cos(s_1) - 2\cos(s_2)$ be the Toeplitz generating function in the case of FE approximations on a square $\Omega = (0,1)^2$ with Friedrichs-Keller meshes, or standard FD discretizations.

Since these two functions are equivalent, i.e.,

$$\frac{\sqrt{3}}{3} f \leq \tilde{f} \leq \sqrt{3} f \quad \text{on } \mathcal{D} = (-\pi, \pi]^2,$$

by virtue of LPO properties, the corresponding Toeplitz matrix sequences are spectrally equivalent, i.e.,

$$\frac{\sqrt{3}}{3} T_n(f) \leq T_n(\tilde{f}) \leq \sqrt{3}\, T_n(f) \quad \text{for any } n.$$

Remark: $\tilde{f}$ is the most natural function from the FE point of view, since no contribution is lost owing to the gradient orthogonality.

Mesh role - Equilateral elements - III

Relationships between the functions f and $\tilde{f}$ can be fully exploited in performing the spectral analysis. More precisely, we have that

$$\begin{aligned} \lambda_{\min}(\Theta_n(1)) &\geq \lambda_{\min}(T_N(\tilde{f})) \geq \frac{\sqrt{3}}{3}\lambda_{\min}(T_N(f)) \geq c_1 h_N^2, \\ \lambda_{\min}(\Theta_n(1)) &\leq \lambda_{\min}(T_{\tilde{N}}(\tilde{f})) \leq \sqrt{3}\,\lambda_{\min}(T_{\tilde{N}}(f)) \leq c_2 h_{\tilde{N}}^2 \end{aligned}$$

In addition, the same matrix Π can also be considered in the more general setting, i.e.

$$\{A_n(a)\} = \{\Pi\, A_N(a)\, \Pi^T\}$$

since the key point is that each internal node in Ω is a vertex of the some constant number of triangles.

By referring to projection arguments, the spectral analysis can be equivalently performed both on the matrix sequence $\{A_n(a)\}$ and $\{A_N(a)\}$.

Mesh role - Equilateral elements - III

Relationships between the functions f and $\tilde{f}$ can be fully exploited in performing the spectral analysis. More precisely, we have that

$$\begin{aligned} \lambda_{\min}(\Theta_n(1)) &\geq \lambda_{\min}(T_N(\tilde{f})) \geq \frac{\sqrt{3}}{3}\lambda_{\min}(T_N(f)) \geq c_1 h_N^2, \\ \lambda_{\min}(\Theta_n(1)) &\leq \lambda_{\min}(T_{\tilde{N}}(\tilde{f})) \leq \sqrt{3}\,\lambda_{\min}(T_{\tilde{N}}(f)) \leq c_2 h_{\tilde{N}}^2 \end{aligned}$$

In addition, the same matrix Π can also be considered in the more general setting, i.e.

$$\{A_n(a)\} = \{\Pi\, A_N(a)\, \Pi^T\}$$

since the key point is that each internal node in Ω is a vertex of the some constant number of triangles.

By referring to projection arguments, the spectral analysis can be equivalently performed both on the matrix sequence $\{A_n(a)\}$ and $\{A_N(a)\}$.

The Preconditioning Strategy - Definition

Let $\{A_n(a,\vec{\beta})\}$, $n = n(h)$ be the matrix sequence associated to a family $\{\mathcal{T}_h\}$, with decreasing parameter h.

The considered preconditioning matrix sequence, proposed in [1], is defined as

$$\{P_n(a)\}, \quad P_n(a) = D_n^{\frac{1}{2}}(a)\Theta_n(1)D_n^{\frac{1}{2}}(a)$$

where $D_n(a) = \mathrm{diag}(\Theta_n(a))\mathrm{diag}^{-1}(\Theta_n(1))$, i.e., the suitable scaled main diagonal of $\Theta_n(a)$, where $\Theta_n(a)$ equals $A_n(a,0)$.

Remark: the preconditioner is tuned only with respect to the diffusion matrix $\Theta_n(a)$ since we are assuming that the convection phenomenon is not dominant, and no stabilization is required in order to avoid spurious oscillations into the solution.

[1] Serra-Capizzano, Numer. Math., 1999.

The Preconditioning Strategy - Definition

Let $\{A_n(a,\vec{\beta})\}$, $n = n(h)$ be the matrix sequence associated to a family $\{\mathcal{T}_h\}$, with decreasing parameter h.

The considered preconditioning matrix sequence, proposed in [1], is defined as

$$\{P_n(a)\}, \quad P_n(a) = D_n^{\frac{1}{2}}(a)\Theta_n(1)D_n^{\frac{1}{2}}(a)$$

where $D_n(a) = \mathrm{diag}(\Theta_n(a))\mathrm{diag}^{-1}(\Theta_n(1))$, i.e., the suitable scaled main diagonal of $\Theta_n(a)$, where $\Theta_n(a)$ equals $A_n(a,0)$.

Remark: the preconditioner is tuned only with respect to the diffusion matrix $\Theta_n(a)$ since we are assuming that the convection phenomenon is not dominant, and no stabilization is required in order to avoid spurious oscillations into the solution.

[1] Serra-Capizzano, Numer. Math., 1999.

Spectral Analysis - I

We analyze the spectral properties of the preconditioned matrix sequences

$\{P_n^{-1}(a)\Theta_n(a)\}$ wrt PCG

$\{P_n^{-1}(a)\mathrm{Re}(A_n(a,\vec{\beta}))\}$, $\{P_n^{-1}(a)\mathrm{Im}(A_n(a,\vec{\beta}))\}$, $\{P_n^{-1}(a)A_n(a,\vec{\beta})\}$ wrt PGMRES

in the special case of structured uniform mesh sequences on the hexagonal and on the square domain.

Aim:

- to quantify the difficulty of the linear system resolution vs the accuracy of the approximation scheme;
- to prove the effectiveness/optimality of the preconditioned iterative method.

Spectral Analysis - II

In the case of the considered FE approximation of Convection-Diffusion Eqns, the Hermitian/skew-Hermitian splitting is given by

$$\mathrm{Re}(A_n(a,\vec{\beta})) = \sum_{K\in\mathcal{T}_h} \mathrm{Re}(A_n^K(a,\vec{\beta})) = \Theta_n(a) + \mathrm{Re}(\Psi_n(\vec{\beta})) \ \text{ spd},$$

$$\mathrm{i}\,\mathrm{Im}(A_n(a,\vec{\beta})) = \mathrm{i} \sum_{K\in\mathcal{T}_h} \mathrm{Im}(A_n^K(a,\vec{\beta})) = \mathrm{i}\,\mathrm{Im}(\Psi_n(\vec{\beta})),$$

and can be performed on any single elementary matrix related to $\mathcal{T}_h$.
Notice that $\mathrm{Re}(\Psi_n(\vec{\beta})) = 0$ if $\mathrm{div}(\vec{\beta}) = 0$.

Lemma

Let

$$\{E_n(\vec{\beta})\}, \quad E_n(\vec{\beta}) := \mathrm{Re}(\Psi_n(\vec{\beta})).$$

Under the regularity assumptions, then it holds

$$\|E_n(\vec{\beta})\|_2 \le \|E_n(\vec{\beta})\|_\infty \le Ch^2,$$

with C absolute constant only depending on $\vec{\beta}(\mathbf{x})$ and Ω.

Spectral Analysis - Diffusion Eqns

Theorem

Let $\{\Theta_n(a)\}$ and $\{P_n(a)\}$ be the Hermitian positive definite matrix sequences previously defined.
Under the regularity assumptions, the sequence $\{P_n^{-1}(a)\Theta_n(a)\}$ is properly clustered at 1.
Moreover, for any n all the eigenvalues of $P_n^{-1}(a)\Theta_n(a)$ belong to an interval $[d, D]$ well separated from zero (Spectral equivalence property).

The previous results prove the optimality of the PCG method.

The proof technique refers to a previously analyzed FD case [1] and makes use of the equivalence of the Toeplitz generating functions.

[1] Serra-Capizzano, TP, ETNA, 2000; SIMAX 2003.

Spectral Analysis - Diffusion Eqns

Theorem

Let $\{\Theta_n(a)\}$ and $\{P_n(a)\}$ be the Hermitian positive definite matrix sequences previously defined.
Under the regularity assumptions, the sequence $\{P_n^{-1}(a)\Theta_n(a)\}$ is properly clustered at 1.
Moreover, for any n all the eigenvalues of $P_n^{-1}(a)\Theta_n(a)$ belong to an interval $[d, D]$ well separated from zero (Spectral equivalence property).

The previous results prove the optimality of the PCG method.

The proof technique refers to a previously analyzed FD case [1] and makes use of the equivalence of the Toeplitz generating functions.

---

[1] Serra-Capizzano, TP, ETNA, 2000; SIMAX 2003.

Spectral Analysis - Convection-Diffusion Eqns - I

Theorem

Let $\{\mathrm{Re}(A_n(a,\vec{\beta}))\}$ and $\{P_n(a)\}$ be the Hermitian positive definite matrix sequences previously defined.
Under the regularity assumptions, the sequence $\{P_n^{-1}(a)\mathrm{Re}(A_n(a,\vec{\beta}))\}$ is properly clustered at 1.
Moreover, for any n all the eigenvalues of $P_n^{-1}(a)\mathrm{Re}(A_n(a,\vec{\beta}))$ belong to an interval $[d,D]$ well separated from zero (Spectral equivalence property).

The proof technique refers to a previously analyzed FD case [1].
It is extended for dealing with the contribution given by $E_n(\vec{\beta})$ and makes use of the equivalence of the Toeplitz generating functions.

[1] Serra-Capizzano, TP, ETNA, 2000; SIMAX 2003.

Spectral Analysis - Convection-Diffusion Eqns - I

Theorem

Let $\{\mathrm{Re}(A_n(a,\vec{\beta}))\}$ and $\{P_n(a)\}$ be the Hermitian positive definite matrix sequences previously defined.
Under the regularity assumptions, the sequence $\{P_n^{-1}(a)\mathrm{Re}(A_n(a,\vec{\beta}))\}$ is properly clustered at 1.
Moreover, for any n all the eigenvalues of $P_n^{-1}(a)\mathrm{Re}(A_n(a,\vec{\beta}))$ belong to an interval $[d, D]$ well separated from zero (Spectral equivalence property).

The proof technique refers to a previously analyzed FD case [1].
It is extended for dealing with the contribution given by $E_n(\vec{\beta})$ and makes use of the equivalence of the Toeplitz generating functions.

[1] Serra-Capizzano, TP, ETNA, 2000; SIMAX 2003.

Spectral Analysis - Convection-Diffusion Eqns - II

Theorem

Let $\{\mathrm{Im}(A_n(a,\vec{\beta}))\}$ and $\{P_n(a)\}$ be the Hermitian matrix sequences previously defined.
Under the regularity assumptions, the sequence $\{P_n^{-1}(a)\mathrm{Im}(A_n(a,\vec{\beta}))\}$ is spectrally bounded and properly clustered at 0 with respect to the eigenvalues.

The proof technique refers to the spectral Toeplitz theory and to the standard FE assembling procedure, according to a more natural local domain analysis approach [1].

[1] Beckermann, Serra-Capizzano, SINUM, 2007.

Spectral Analysis - Convection-Diffusion Eqns - II

Theorem

Let $\{\mathrm{Im}(A_n(a,\vec{\beta}))\}$ *and* $\{P_n(a)\}$ *be the Hermitian matrix sequences previously defined.*
Under the regularity assumptions, the sequence $\{P_n^{-1}(a)\mathrm{Im}(A_n(a,\vec{\beta}))\}$ *is spectrally bounded and properly clustered at* 0 *with respect to the eigenvalues.*

The proof technique refers to the spectral Toeplitz theory and to the standard FE assembling procedure, according to a more natural local domain analysis approach [1].

[1] Beckermann, Serra-Capizzano, SINUM, 2007.

Spectral Analysis - Convection-Diffusion Eqns - III

On the basis of these two splitted spectral results, we can easily obtain the spectral description of the whole preconditioned matrix sequence $\{P_n^{-1}(a)A_n(a,\vec{\beta})\}$.

Theorem

Let $\{(A_n(a,\vec{\beta}))\}$ and $\{P_n(a)\}$ be the matrix sequences previously defined. Under the regularity assumptions, the sequence $\{P_n^{-1}(a)A_n(a,\vec{\beta})\}$ is properly clustered at $1 \in \mathbf{C}^+$ with respect to the eigenvalues. In addition, these eigenvalues all belong to a uniformly bounded rectangle with positive real part, well separated from zero.

The proof technique refers to an analogous result in the case of FD discretizations of Convection-Diffusion Eqns [1] by making use of the field of values properties and of the results claimed in the last two theorems.

[1] Bertaccini, Golub, Serra-Capizzano, SIMAX, 2007.

Spectral Analysis - Convection-Diffusion Eqns - III

On the basis of these two splitted spectral results, we can easily obtain the spectral description of the whole preconditioned matrix sequence $\{P_n^{-1}(a)A_n(a,\vec{\beta})\}$.

Theorem

Let $\{(A_n(a,\vec{\beta}))\}$ and $\{P_n(a)\}$ be the matrix sequences previously defined. Under the regularity assumptions, the sequence $\{P_n^{-1}(a)A_n(a,\vec{\beta})\}$ is properly clustered at $1 \in \mathbf{C}^+$ with respect to the eigenvalues. In addition, these eigenvalues all belong to a uniformly bounded rectangle with positive real part, well separated from zero.

The proof technique refers to an analogous result in the case of FD discretizations of Convection-Diffusion Eqns [1] by making use of the field of values properties and of the results claimed in the last two theorems.

[1] Bertaccini, Golub, Serra-Capizzano, SIMAX, 2007.

Spectral Analysis - Remarks

Remark

The previous Lemma and Theorems hold both in the case in which the matrix elements are evaluated exactly and whenever a quadrature formula with error $O(h^2)$ is considered to approximate the involved integrals.

Numerical Tests

- All the reported numerical experiments are performed in Matlab.
- The iterative solvers starts with zero initial guess.
- The stopping criterion is $||r_k||_2 \leq 10^{-7}||r_0||_2$.
- The PGMRES is applied without restart.

The numerical tests analyze the effectiveness of the preconditioning strategy in the case of the following diffusion coefficient functions:

- Test I: $a(x, y) = a_1(x, y) = \exp(x + y)$
- Test II: $a(x, y) = a_2(x, y) = \exp(x + |y - 1/2|^{3/2})$
- Test III: $a(x, y) = a_3(x, y) = \exp(x + |y - 1/2|)$

The convection coefficient is $\vec{\beta}(x, y) = [x \;\; y]^T$ (diffusion dominated problem).

Numerical Tests - PCG

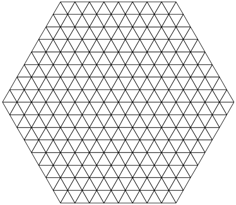
Structured/unstructured meshes on the hexagonal domain

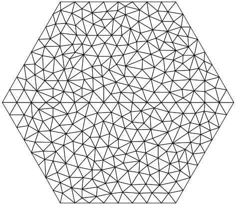
n	$a_1(x,y)$	$a_2(x,y)$	$a_3(x,y)$
37	3	4	5
169	3	4	5
721	3	4	4
2977	3	4	4
12095	3	4	4
48769	3	4	4

# of iterations in the case of structured meshes

n	$a_1(x,y)$	$a_2(x,y)$	$a_3(x,y)$
28	5	5	5
73	4	4	5
265	4	4	5
1175	4	4	5
4732	4	4	5
19288	4	4	5

# of iterations in the case of unstructured meshes





Numerical Tests - PGMRES

Structured/unstructured meshes on the hexagonal domain

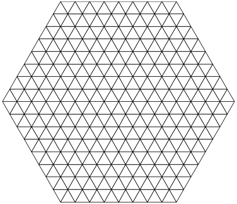
$\vec{\beta}(x,y) = [x\ y]^T$ convection coefficient.

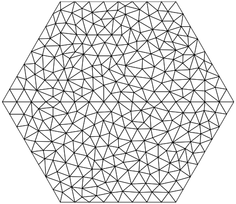
n	$a_1(x,y)$	$a_2(x,y)$	$a_3(x,y)$
37	4	5	5
169	4	5	5
721	4	5	5
2977	4	5	5
12095	4	5	5
48769	4	5	5

# of iterations in the case of structured meshes

n	$a_1(x,y)$	$a_2(x,y)$	$a_3(x,y)$
28	4	5	5
73	4	5	5
265	4	5	5
1175	4	5	5
4732	4	5	5
19288	4	5	5

# of iterations in the case of unstructured meshes





Numerical Tests - PCG

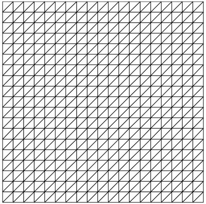
Structured/unstructured meshes on the square domain

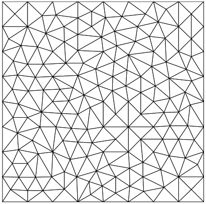
n	$a_1(x,y)$	$a_2(x,y)$	$a_3(x,y)$
81	3	4	4
3611	3	4	5
1521	3	4	5
6241	3	4	5
25281	3	4	5

# of iterations in the case of structured meshes

n	$a_1(x,y)$	$a_2(x,y)$	$a_3(x,y)$
142	4	5	5
725	4	5	5
1538	4	5	5
7510	4	5	5
15690	4	5	5

# of iterations in the case of unstructured meshes





Numerical Tests - PGMRES

Structured/unstructured meshes on the square domain

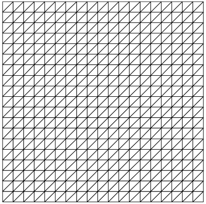
$\vec{\beta}(x,y) = [x\ y]^T$ convection coefficient.

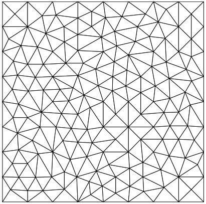
n	$a_1(x,y)$	$a_2(x,y)$	$a_3(x,y)$
81	4	5	5
3611	5	5	6
1521	5	6	6
6241	5	6	6
25281	5	6	6

# of iterations in the case of structured meshes

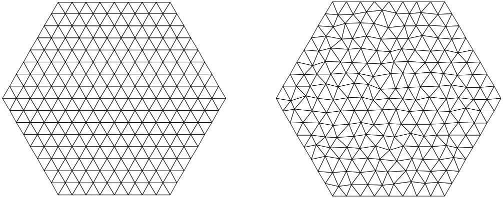
n	$a_1(x,y)$	$a_2(x,y)$	$a_3(x,y)$
142	4	5	5
725	5	5	6
1538	5	6	6
7510	5	6	6
15690	5	6	7

# of iterations in the case of unstructured meshes





Numerical Tests



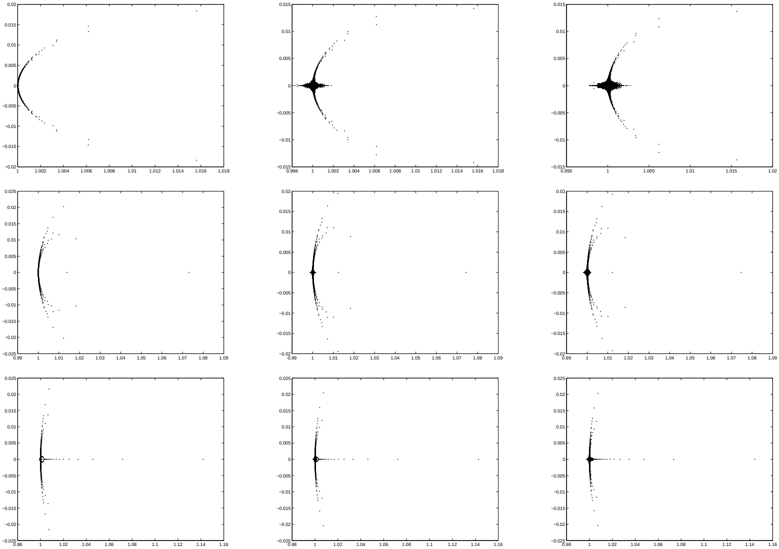
PCG		
n	P	$\tilde{P}$
37	4	9
169	4	10
721	4	11
2977	4	12
12095	4	12

PGMRES		
n	P	$\tilde{P}$
37	4	8
169	4	8
721	4	9
2977	4	11
12095	4	11

$$P_n(a) = D_n^{\frac{1}{2}}(a) A_n(1,0) D_n^{\frac{1}{2}}(a)$$
$$\tilde{P}_n(a) = D_n^{\frac{1}{2}}(a) \tilde{T}_n D_n^{\frac{1}{2}}(a),$$
$$\tilde{T}_n = \Pi T_m \Pi,$$
$$T_m = T_m(6 - 2\cos(s) - 2\cos(t) - 2\cos(s+t))$$

Number of PCG and PGMRES iterations - structured and unstructured meshes, $a_1(x,y)$, $\vec{\beta}(x,y) = [x\ y]^T$.

Numerical Tests



Complexity Issues - I

Definition (Axelsson, Neytcheva, 1994)

Let $\{A_m \mathbf{x}_m = \mathbf{b}_m\}$ be a given sequence of linear systems of increasing dimensions. An iterative method is *optimal* if

- the arithmetic cost of each iteration is at most proportional to the complexity of a matrix-vector product with matrix A_m,
- the number of iterations for reaching the solution within a fixed accuracy can be bounded from above by a constant independent of m.

Since

$$P_n(a) = D_n^{1/2}(a)\Theta_n(1)D_n^{1/2}(a),$$

the solution of FE linear system with matrix $A_n(a, \vec{\beta})$ is reduced to computations involving diagonals and the matrix $\Theta_n(1)$.

Complexity Issues - II

The latter task can be efficiently performed by means of fast Poisson solvers (e.g. cyclic reduction idea [1]) or several specialized algebraic multigrid methods [2] or geometric multigrid methods [3].

Therefore, for structured uniform meshes and under the regularity assumptions, the optimality of the PCG method is theoretically proved.
Favorable convergence properties are expected for the PGMRES method.

The numerical performances do not worsen in the case of unstructured meshes.

[1] Buzbee, Dorr, George, Golub, SINUM, 1971.
[2] Serra-Capizzano, Numer. Math., 2002.
[3] Trottenberg, Oosterlee, Schüller, Academic Press, 2001.

Open Problems & Conclusions

It is self-evident that the problem at hand is just an academic example. However, it is a fact that a good mesh generator will locally produce a partitioning, which is "asymptotically" similar to the considered one.
The latter fact has a practical important counterpart since the academic preconditioner $\tilde{P}_n(a)$ is optimal for the real case with nonconstant coefficients and with the unstructured partitioning.
A theoretical ground supporting these observation is still missing and would be worth in our opinion to be studied and developed.

Moreover, taking into account all these results, it will be interesting to devise suitable preconditioning strategies for

- piecewise constant diffusion coefficient problems,
- anisotropic problems,
- convection dominated problem stabilized by streamline artificial diffusion.